STABLE RANDOM FIELDS

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This thesis concentrates on the extreme value theory of measurable stationary symmetric non-Gaussian stable random fields and its connection to the length of memory for both discrete and continuous parameter cases.

Firstly, we establish a connection between the structure of a stationary symmetric α-stable ($0 < \alpha < 2$) discrete parameter random field $\{X_t\}_{t\in\mathbb{Z}^d}$ and ergodic theory of non-singular $\mathbb{Z}^d$-actions, elaborating on a previous work by Rosiński (2000). With the help of this connection, we study the sequence of extreme values $M_n := \max_{\|t\|_\infty \le n,\, t\ge 0} |X_t|$ of the field. Here, $t \geq 0$ means all the co-ordinates of t are nonnegative. Depending on the ergodic theoretical and group theoretical structures of the underlying $\mathbb{Z}^d$-action, we observe different kinds of asymptotic behavior of this sequence of extreme values. In the discrete-parameter case, we also consider the point process sequence $\{ \sum_{\|t\|_\infty \le n} \delta_{b_n^{-1} X_t} : n \geq 1\}$ for a suitable choice of scaling sequence $b_n \uparrow \infty$. If the random field is generated by a dissipative $\mathbb{Z}^d$-action then this point process sequence converges weakly to a cluster Poisson process with $b_n = n^{d/\alpha}$. For the conservative case, we look at a specific class of stable random fields for which the exact effective dimension $p \leq d$ is known. For this class of random fields, using $b_n = n^{p/\alpha}$ and normalizing the point process itself we get, as weak limit, a random measure which is not a point process.

For the continuous-parameter case, we first develop the notions of conservativity and dissipativity of nonsingular $\mathbb{R}^d$ actions and use them to obtain the continuous-parameter analogues of the structure results presented in the discrete case with the assumption that the random field $\{X_t\}_{t\in\mathbb{R}^d}$ is measurable. We also observe that any stationary measurable

random field is continuous in probability, which is then applied to compute the rate of growth of the maxima of the random field as the parameter t takes values in increasing boxes. As in the discrete parameter case, we notice different rates of growth of the maxima depending on the ergodic theoretic nature of the underlying $\mathbb{R}^d$-action. It can be argued, in both discrete and continuous-parameter cases, that this change of rate is governed by the length of memory of the field.

BIOGRAPHICAL SKETCH

Parthanil Roy was born on October 23, 1978 in Calcutta (presently Kolkata), India. In June 1997 he graduated from the Hindu School, Kolkata and joined the Indian Statistical Institute, where he was introduced to the wonderful world of mathematics and statistics. He received his B. Stat (Bachelor of Statistics) degree in June 2000 and his M. Stat (Master of Statistics) degree in June 2002 from the Indian Statistical Institute.

In August 2002, he joined Cornell University, Ithaca, New York, USA as a graduate student in the School of Operations Research and Industrial Engineering (now known as the School of Operations Research and Information Engineering) with concentration in Applied Probability and Statistics.

Upon completion of his PhD he will spend a year as a post-doctoral fellow in RiskLab, Department of Mathematics, ETH, Zürich, and then join the Department of Statistics and Probability at Michigan State University as an assistant professor.

In memory of Baba

and

To Ma, Dada and Boudi

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Chapter 1

Introduction

Random fields are stochastic processes indexed by multidimensional sets (e.g., $\mathbb{Z}^d$, $\mathbb{R}^d$, manifolds, etc.). They arise in modeling of any kind of spatial data (e.g., agricultural data, weather data, data on brain mapping, image science, etc.). Many of these data sets show heavy tails in their distributions. Since non-Gaussian stable distributions form an important class of such distributions, stable random fields become very useful for modeling spatial data with heavy tails. In this thesis, we will consider both discrete and continuous-parameter (indexed by $\mathbb{Z}^d$ and $\mathbb{R}^d$ respectively) stationary symmetric non-Gaussian stable random fields and study their properties.

A random variable X is said to follow a symmetric α-stable ($S\alpha S$) distribution with scale parameter $\sigma > 0$ if

$$E(e^{i\theta X}) = e^{-\sigma^\alpha |\theta|^\alpha} \quad \text{for all } \theta \in \mathbb{R}.$$

Here $0 < \alpha \le 2$. When $\alpha = 2$ this reduces to a Gaussian distribution. We will concentrate on the non-Gaussian case, and hence, assume that $0 < \alpha < 2$, unless mentioned otherwise. In this case, α can be regarded as the tail parameter because

$$P(|X| > \lambda) \sim C_\alpha \sigma^\alpha \lambda^{-\alpha} \quad \text{as } \lambda \to \infty\,,$$

where C_α is a constant (depending on α) known as the stable tail constant. In particular, for $0 < \alpha < 2$, we have

$$E|X|^p < \infty \quad \text{for any} \quad 0 < p < \alpha\,,$$
$$E|X|^p = \infty \quad \text{for any} \quad p \ge \alpha\,.$$

See, for example, Feller (1971) and Samorodnitsky and Taqqu (1994) for further reference on $S\alpha S$ distributions and processes.

For $T = \mathbb{Z}$ or $\mathbb{R}$, $\{X_t\}_{t \in T^d}$ is called a symmetric α-stable ($S\alpha S$) random field if for all $c_1, c_2, \ldots, c_k \in \mathbb{R}$, and $t_1, t_2, \ldots, t_k \in T^d$, $\sum_{j=1}^{k} c_j X_{t_j}$ follows a symmetric α-stable distribution. A random field $\{X_t\}_{t \in T^d}$ is called stationary if

$$\{X_t\} \stackrel{d}{=} \{X_{t+s}\} \quad \text{for all } s \in T^d \,.$$

Stationarity means that the law of the random field is invariant under the action of the group of shift transformations on the index-parameter $t \in T^d$.

1.1 Integral Representation and Structure Results

It has been known since Bretagnolle et al. (1966) and Schreiber (1972) that a measurable stationary $S\alpha S$ random field, $\{X_t\}_{t \in T^d}$, has an integral representation of the type

$$X_t \stackrel{d}{=} \int_S f_t(s) M(ds), \quad t \in T^d \,, \tag{1.1.1}$$

where M is a $S\alpha S$ random measure on some standard Borel space $(S, \mathcal{S})$ with σ-finite control measure μ and $f_t \in L^\alpha(S, \mu)$ for all $t \in T^d$. See also Schilder (1970), Kuelbs (1973), and Samorodnitsky and Taqqu (1994). The structure of measurable stationary $S\alpha S$ random fields was studied in detail by Rosiński (1995) for $d = 1$ and Rosiński (2000) for a general $d \geq 1$, where using certian rigidity properties of spaces L^α, $0 < \alpha < 2$, it was established that for such random fields $\{f_t\}$ in (1.1.1) can be chosen to be of the form

$$f_t(s) = c_t(s) \left(\frac{d\mu \circ \phi_t}{d\mu}(s) \right)^{1/\alpha} f \circ \phi_t(s), \quad t \in T^d \,, \tag{1.1.2}$$

where $f \in L^\alpha(S, \mu)$, $\{\phi_t\}_{t \in T^d}$ is a nonsingular group action of the group T^d on (S, μ) (i.e., $(t, s) \mapsto \phi_t(s)$ is a measurable map $T^d \times S \to S$ such that $\phi_{u+v}(s) = \phi_u(\phi_v(s))$, $\phi_0(s) = s$ and $\mu \sim \mu \circ \phi_t$ for all $u, v, t \in T^d$), and $\{c_t\}_{t \in \mathbb{Z}^d}$ is a measurable cocycle for $\{\phi_t\}$ taking values in $\{-1, +1\}$ (i.e., $(t, s) \mapsto c_t(s)$ is a measurable map $T^d \times S \to \{-1, +1\}$ such that for all $u, v \in T^d$, $c_{u+v}(s) = c_v(s)c_u(\phi_v(s))$ for μ-a.a. $s \in S$). Conversely, if $\{f_t\}$ is of the form (1.1.2) then $\{X_t\}$ defined by (1.1.1) is a stationary $S\alpha S$ random field. See, for example, Aaronson (1997), Varadarajan (1970), and Zimmer (1984) for discussions on nonsingular (also known as quasi-invariant) group actions.

Using the integral representation (1.1.1) of the form (1.1.2) a unique in law decomposition of $\{X_t\}$ into three independent stationary $S\alpha S$ random fields was obtained in Rosiński (2000), namely,

$$X_t \stackrel{d}{=} X_t^{(1)} + X_t^{(2)} + X_t^{(3)}, \quad t \in T^d,$$

where $\{X_t^{(1)}\}_{t\in T^d}$ is a superposition of moving averages (the so-called mixed moving average in the terminology of Surgailis et al. (1993)), $\{X_t^{(2)}\}_{t\in T^d}$ is a harmonizable random field, and $\{X_t^{(3)}\}_{t\in T^d}$ is a stationary $S\alpha S$ random field with no mixed moving average or harmonizable component. In the one-dimensional case (Rosiński (1995)) this decomposition was connected to the ergodic theory of nonsingular flows (see, for example, Aaronson (1997) and Krengel (1985)) using Krengel's theorem classifying dissipative flows (see Krengel (1969)). However, this connection was missing in the $d > 1$ case because of the unavailability of Krengel's theorem. In this work, we have been able to remove this obstacle elaborating on Theorem 2.1 in Rosiński (2000). In particular, we have been able to extend Theorems 4.1 and 4.4 in Rosiński (1995) to the $d > 1$ case. In the discrete-parameter case, these results have obvious extensions to the stationary $S\alpha S$ random fields indexed by countable groups.

Using the language of positive-null decomposition of nonsingular flows (see Section 1.4 in Aaronson (1997) and Section 3.4 in Krengel (1985)) another decomposition of measurable stationary $S\alpha S$ processes was obtained in Samorodnitsky (2005) and this decomposition was used to characterize the ergodicity of such a process. Decompositions based on ergodic theory of nonsingular flows were also obtained for self-similar

$S\alpha S$ mixed moving average processes (with stationary increments) in Pipiras and Taqqu (2002a) and Pipiras and Taqqu (2002b). The random field analogues of these decomposition results are still unknown.

1.2 Long Range Dependence and Extreme Value Theory

Long range dependence (also known as long memory), a property observed in many real life processes, refers to dependence between observations X_t far separated in t. Historically, it was first observed by the famous British hydrologist Harold Edwin Hurst, who noticed an empirical phenomenon (now known as the Hurst phenomenon; see Hurst (1951) and Hurst (1955)) while looking at measurements of the water flow in the Nile river. In the 1960s a series of papers of Benoit Mandelbrot and his co-workers tried to explain the Hurst phenomenon using long range dependence. See Mandelbrot and Wallis (1968) and Mandelbrot and Wallis (1969). From then on processes having long memory have been used in many different areas including but not limited to economics, internet modelling, climate studies, linguistics, DNA sequencing, etc. For example, recent statistical data for financial markets and network traffic suggests use of models having long memory. See Lobato and Velasco (2000), and Willinger et al. (2003). For a detailed discussion on long range dependence, see Samorodnitsky (2007) and the references therein.

Surprisingly, very few of these publications gives a formal definition of long range dependence. Even if definitions are given, they vary from author to author. Most of the classical definitions of long range dependence appearing in the literature are based on the second-order properties (e.g., covariances, spectral density, and variances of partial sums, etc.) of stochastic processes mainly because of their simplicity and statistical tractability. For example, one of the most widely accepted definitions of long range dependence for a stationary Gaussian process is the following: we say that a stationary Gaussian process has long range dependence if its correlation function decays slowly enough to make it not summable. In the heavy tails context, however, this definition becomes ambiguous because the correlation function may not even exist in the heavy tails case and even if it exists it may not have enough information about the dependence structure of the process. Covariance-like functions have been tried (see, for example, Astrauskas et al. (1991), Maejima and Yamamoto (2003), and Magdziarz (2005)) but their usefulness seems to be limited.

In the context of stationary $S\alpha S$ processes ($0 < \alpha < 2$), instead of looking for a substitute for correlation function, Samorodnitsky (2004a) suggested a new approach through phase transition phenomena as follows: Suppose that $(P_\theta,\ \theta \in \Theta)$ is a family of laws of a stationary stochastic process, where θ is a parameter of the process lying in a parameter space Θ. If Θ can be partitioned into Θ_0 and Θ_1 in such a way that a significant number of functionals acting on the sample paths of this stochastic process change dramatically as we pass from Θ_0 to Θ_1, then this phase transition can be thought of as a change from short memory to long memory. In the aforementioned papers, the partial maxima of measurable stationary $S\alpha S$ processes and its rate of growth are considered (see Samorodnitsky (2004a) for the discrete parameter case and Samorodnitsky (2004b) for the continuous parameter case) and a transition boundary is observed based on the ergodic theoretical properties of a highly infinite-dimensional parameter of the process, namely the underlying group action $\{\phi_t\}$ as in (1.1.2).

Motivated by the classical extreme value theory and the approach to long range dependence mentioned above, we look at the partial maxima sequence

$$M_n = \max_{\|t\|_\infty \le n,\, t \ge 0} |X_t|\,, \quad n = 1, 2, 3, \ldots$$

for the discrete-parameter case, and the maxima process

$$M_\tau = \sup_{\|t\|_\infty \le \tau,\, t \ge 0} |X_t|, \quad \tau > 0$$

for the continuous-parameter case with the assumption that the random field is locally bounded (i.e., $M_\tau < \infty$ with probability 1 for all $\tau > 0$). Here $t \ge 0$ means all the coordinates of t are nonnegative. In the continuous-parameter case, we first establish that any stationary measurable random field is continuous in probability and then use a separable version of the field to define M_τ without running into any measurability problem. This continuity result follows from Banach's theorem for Polish groups (see Section 1.6 in Aaronson (1997)). In the $d = 1$ case, the corresponding result for measurable processes with stationary increments was proved by Surgailis et al. (1998) using a result of Cohn (1972).

We estimate the rate of growth of M_n and M_τ as well as compute their scaling limit whenever the exact rate of growth is known. In the one-dimensional case, similar results are presented for certain Gaussian processes and diffusion processes in Leadbetter et al. (1983) and Berman (1992). Some more recent general results can be found in Albin (1990). Following the arguments in Samorodnitsky (2004a) and Samorodnitsky (2004b) and using the structure results of $S\alpha S$ random fields, we get that the above partial maxima grow at a rate $n^{d/\alpha}$ ($\tau^{d/\alpha}$ in the continuous parameter case) if the underlying group action is not conservative (see, for example, Aaronson (1997)) and in this case the scaling limit of the maxima happens to be a Frechet-type extreme value random variable as in the iid case. Hence this case can be regarded as the short memory case. On the other hand, when the group action $\{\phi_t\}$ in (1.1.2) is conservative, in general, we can only prove the maxima to grow slower than $n^{d/\alpha}$ ($\tau^{d/\alpha}$ in the continuous parameter case). This slow growth rate of this partial maxima can be explained by the presence of long range dependence in the random field in parallel to Slepian's Lemma (see Slepian (1962)) in the Gaussian case. These results are proved without referring to Maharam's Theorem (see Maharam (1964)), which is unavailable in the $d \geq 2$ case.

For stationary $S\alpha S$ discrete-parameter random fields generated by conservative actions, we make further investigations on the actual rate of growth of the partial maxima sequence M_n using the theory of finitely generated abelian groups (see, for example, Lang (2002)) together with counting of the number of lattice points in dilates of rational polytopes (see De Loera (2005)). In general, it is easy to observe as in the $d = 1$ case, Samorodnitsky (2004a), that this rate of growth can be pretty much anything slower than $n^{d/\alpha}$ depending on the underlying action and the function f. In some cases, viewing the action as a group of nonsingular transformations and studying the algebraic structure of this group we can obtain the effective dimension $p \leq d$ of the random field, which gives us better ideas about the rate of growth of the partial maxima as well as the length of memory of the random field.

1.3 Connections to Point Processes

A point process is a random distribution of points in space. Formally, it can defined as a random element in the space of Radon point measures on some locally compact topological space E (the space where the points live) with a countable base. For a thorough understanding of many structural results in extreme value theory, knowledge of point processes in essential. Weak convergence of point processes along with a clever use of the continuous mapping theorem is a very widely accepted and immensely useful technique to prove various limit theorems for extremes and other functionals of a stochastic process. See, for example, Resnick (1987) for a discussion on point processes and their use in extreme value theory. See also Neveu (1977) and Kallenberg (1983).

We are interested in the weak convergence of the sequence of point processes on

$E = [-\infty, \infty] - \{0\}$

$$N_n = \sum_{\|t\|_\infty \le n} \delta_{b_n^{-1} X_t}, \quad n = 1, 2, 3, \ldots \tag{1.3.1}$$

induced by the discrete-parameter stationary $S\alpha S$ random field $\{X_t\}$ with an appropriate choice of scaling sequence $b_n \uparrow \infty$. Here δ_x denotes the point mass at x. For $d = 1$, this was considered by Resnick and Samorodnitsky (2004) and it was shown that the point process sequence can have diverse behavior depending on the ergodic theoretical nature of of the underlying group action. When the action is dissipative (see Aaronson (1997)) the point process sequence (1.3.1) converges to a cluster Poisson process with $b_n = n^{1/\alpha}$, whereas no general result is known in the one-dimensional case when the action is conservative. Cluster Poisson processes were obtained as weak limits also for the point processes induced by stationary stochastic processes with the marginal distribution having regularly varying tail probabilities as long as the the stationary process satisfies some mild mixing conditions (see Davis and Hsing (1995)) or it is a moving average (see Davis and Resnick (1985)). See also Mori (1977) for various possible weak limits of a two-dimensional point process induced by strong mixing sequences.

In the general case $d \ge 1$, the random field generated by dissipative actions show the exact same behavior as in the one-dimensional case, namely, N_n converges to a cluster Poisson process with $b_n = n^{d/\alpha}$. This limiting behavior of the point process even when the dependence structure is no longer weak or local reflects the fact that the memory of the random field is short. For the long memory case, (i.e., when the underlying group action is conservative) we can comment on the limiting behavior of N_n provided the exact intrinsic dimension $p \le d$ of the field is known so that we can use a scaling sequence $b_n = n^{p/\alpha}$. Long memory of the random field leads to clustering of observations with really large cluster size. Hence, we need to normalize the point process itself in order to ensure weak convergence. (See also Example 4.2 in Resnick and Samorodnitsky (2004).) This normalized point process sequence can be shown to converge weakly to a random measure which is not a point process using group theory and some basic counting arguments. In other words, we observe a phase transition phenomenon for the point process sequence which can also be regarded as a transition from short memory to long memory.

1.4 Outline of Dissertation

As mentioned earlier, we study the properties of stationary symmetric α-stable ($0 < \alpha < 2$) random fields in this thesis. In Chapter 2 and 3 we discuss the discrete-parameter fields and the continuous-parameter case is considered in Chapter 4.

In Chapter 2 we establish a connection between the structure of a stationary symmetric α-stable discrete-parameter random field and ergodic theory of non-singular group actions, elaborating on a previous work by Rosiński (2000). With the help of this connection, we study the extreme values of the field over increasing boxes. Depending on the ergodic theoretical and group theoretical structures of the underlying action, we observe different kinds of asymptotic behavior of this sequence of extreme values.

Chapter 3 deals with the point process (1.3.1) induced by the random field and its weak convergence. We consider two cases depending on whether the underlying group action is dissipative or conservative. In the dissipative case we establish that the point process converges to a cluster Poisson process following verbatim the proof in the $d = 1$ case in Resnick and Samorodnitsky (2004). Due to longer memory in the conservative case, the points cluster so much that we need to normalize the point process itself in order to ensure weak convergence. This normalized point process sequence converges weakly to a random measure but not a point process provided we know the exact effective dimension of the random field.

We first develop the theory of nonsingular $\mathbb{R}^d$-actions in Chapter 4 and then use it to describe the structure of measurable stationary symmetric α-stable continuous-parameter random fields in parallel to the discrete-parameter case. In this chapter, we also estimate the rate of growth of the extreme values of the field over increasing hypercubes.

Chapter 2

Discrete Parameter Fields

2.1 Introduction

In this chapter we study the structure of stationary symmetric α-stable discrete-parameter non-Gaussian random fields and their long range dependence. Recall that a random field $\{X_t\}_{t\in\mathbb{Z}^d}$ is called a symmetric α-stable ($S\alpha S$) random field if for all $c_1, c_2, \ldots, c_k \in \mathbb{R}$, and $t_1, t_2, \ldots, t_k \in \mathbb{Z}^d$, $\sum_{j=1}^k c_j X_{t_j}$ follows a symmetric α-stable distribution. In this chapter we will concentrate on the non-Gaussian case, and hence, we will assume $0 < \alpha < 2$, unless mentioned otherwise. For further reference on $S\alpha S$ distributions and processes the reader is recommended to read Samorodnitsky and Taqqu (1994). A random field $\{X_t\}_{t\in\mathbb{Z}^d}$ is called stationary if

$$\{X_t\} \stackrel{d}{=} \{X_{t+s}\} \quad \text{for all } s \in \mathbb{Z}^d\ . \tag{2.1.1}$$

Stationarity means that the law of the random field is invariant under the action of the group of shift transformations on the index-parameter $t \in \mathbb{Z}^d$.

Our first task is to establish a connection between the ergodic theory of nonsingular $\mathbb{Z}^d$-actions (see Section 1.6 of Aaronson (1997)) and $S\alpha S$ random fields. Using the language of the Hopf decomposition of nonsingular flows a decomposition of stationary $S\alpha S$ processes was established in Rosiński (1995). For a general $d > 1$ a similar decomposition of $S\alpha S$ random fields into independent components was given in Rosiński (2000). We show the connection between this decomposition and ergodic theory. This is done in Section 2.3, without referring to the Chacon-Ornstein theorem, which is unavailable in the case $d > 1$.

More generally, if $(G, +)$ is a countable abelian group with identity element 0, then a random field $\{X_t\}_{t \in G}$ is called G-stationary if (2.1.1) holds for all $s \in G$. Most of the structure results in Section 2.3 have immediate analogs for G-stationary fields. We will present these in details in Section 2.4. Even though our main interest lies with $\mathbb{Z}^d$-indexed random fields, at a certain point in this chapter a more general group structure will become important.

We use the connection with ergodic theory to study the rate of growth of the partial maxima sequence $\{M_n\}$ of the random field $\{X_t\}$ as t runs over a d-dimensional hypercube with an increasing edge length n. In the case $d = 1$ it has been shown in Samorodnitsky (2004a) that this rate drops from $n^{1/\alpha}$ to something smaller as the flow generating the process changes from dissipative to conservative. One can argue that this phase transition qualifies as a transition between short and long memory. In this chapter we establish a similar phase transition result for a general $d \geq 1$.

In Section 2.5, we discuss the asymptotic behavior of a certain deterministic sequence which controls the size of the partial maxima sequence $\{M_n\}$. The treatment here is different from the one-dimensional case due to unavailability of Maharam's theorem (see Maharam (1964)) in the case $d > 1$. In Section 2.6 we calculate the rate of growth of partial maxima of the random field. We show that the rate of growth of M_n is equal to $n^{d/\alpha}$ if the group action has a nontrivial dissipative component, and is strictly smaller than that otherwise.

We discuss connections with the group theoretical properties of the action in Section 2.7. For $S\alpha S$ random fields generated by conservative actions, we view the underlying action as a group of nonsingular transformations and study the algebraic structure of this group to get better estimates on the rate of growth of the partial maxima. Examples illustrating how the maxima of a random field can grow are discussed in Section 2.8.

2.2 Some Ergodic Theory

The details on the notions introduced in this section can be found, for example, in Aaronson (1997). Unless stated otherwise, the statements about sets (e.g., equality or disjointness of two sets) are understood as holding up to a set of measure zero with respect to the underlying measure and all the groups are assumed to be abelian.

Suppose $(S, \mathcal{S}, \mu)$ is a σ-finite standard measure space and $(G, +)$ is a countable group with identity element 0. A collection of measurable maps $\phi_t : S \to S,\ t \in G$ is called a group action of G on S if

1. ϕ_0 is the identity map on S, and
2. $\phi_{u+v} = \phi_u \circ \phi_v$ for all $u, v \in G$.

A group action $\{\phi_t\}_{t\in G}$ of G on S is called nonsingular if $\mu \circ \phi_t \sim \mu$ for all $t \in G$.

A set $W \in \mathcal{S}$ is called a wandering set for the action $\{\phi_t\}_{t\in G}$ if $\{\phi_t(W) :\ t \in G\}$ is a pairwise disjoint collection. The following result (see Proposition 1.6.1 of Aaronson (1997)) gives a decomposition of S into two disjoint and invariant parts.

Proposition 2.2.1. *Suppose G is a countable group and $\{\phi_t\}$ is a nonsingular action of G on S. Then $S = \mathcal{C} \cup \mathcal{D}$ where $\mathcal{C}$ and $\mathcal{D}$ are disjoint and invariant measurable sets such that*

1. $\mathcal{D} = \bigcup_{t\in G} \phi_t(W_*)$ *for some wandering set W_*,*
2. *$\mathcal{C}$ has no wandering subset of positive measure.*

$\mathcal{D}$ is called the dissipative part, and $\mathcal{C}$ the conservative part of the action. The action $\{\phi_t\}$ is called conservative if $S = \mathcal{C}$ and dissipative if $S = \mathcal{D}$.

An action $\{\phi_t\}_{t\in G}$ is free if $\mu(\{s \in S : \phi_t(s) = s\}) = 0$ for all $t \in G - \{0\}$. Note that this definition makes sense because $(S, \mathcal{S})$ is a standard Borel space and hence $\{s \in S : \phi_t(s) = s\} \in \mathcal{S}$. The following result is a version of Halmos' Recurrence Theorem for a nonsingular action of a countable group.

Proposition 2.2.2. *Let $\{\phi_t\}$ be a nonsingular action of a countable group G. If $A \in \mathcal{S}$ and $A \subseteq \mathcal{C}$, then*

$$\sum_{t\in G} I_A \circ \phi_t = \infty \text{ a.e. on } A.$$

Proof. Define

$$F := \{s \in S : \text{ there exists } t \in G, t \neq 0 \text{ such that } \phi_t(s) = s\}.$$

Observe that F is $\{\phi_t\}$-invariant. Restrict $\{\phi_t\}$ to $S - F$. Let $\mathcal{C}_1$ be the conservative part of the restriction. It is easy to observe that $A \cap F^c \subseteq \mathcal{C}_1$ for all $A \subseteq \mathcal{C}$. Since the restricted action is free by Proposition 1.6.2 of Aaronson (1997), we have

$$\sum_{t\in G} I_A \circ \phi_t \geq \sum_{t\in G} I_{A\cap F^c} \circ \phi_t = \infty \text{ a.e. on } A \cap F^c.$$

Clearly,

$$\sum_{t\in G} I_A \circ \phi_t = \infty \text{ a.e. on } A \cap F.$$

This completes the proof. □

Recall that the dual operator of a nonsingular transformation T on S is a linear operator $\hat{T}$ on $L^1(S, \mu)$ such that

$$\int_S \hat{T} f.g d\mu = \int_S f.g \circ T d\mu \quad \text{for all } f \in L^1(\mu) \text{ and } g \in L^\infty(\mu).$$

In particular, if T is invertible, then

$$\hat{T} f = \frac{d\mu \circ T^{-1}}{d\mu} f \circ T^{-1} \quad \text{for all } f \in L^1(\mu)\,,$$

see Section 1.3 in Aaronson (1997). The following proposition, which is an extension of Theorem 1.6.3 of Aaronson (1997) to not necessarily measure-preserving transformations, gives a description of the conservative part of a nonsingular group action $\{\phi_t\}_{t\in G}$ in terms of the operators $\hat{\phi}_t,\ t \in G$ for a countable group G.

Proposition 2.2.3. *If G is a countable group and $\{\phi_t\}$ is a nonsingular action of G on S then for all $f \in L^1(\mu),\ f > 0$,*

$$\mathcal{C} = \{s \in S\,:\, \sum_{t\in G} \hat{\phi}_t f(s) = \infty\}\,.$$

Proof. Fix $f \in L^1(\mu),\ f > 0$. We will first establish that

$$[\sum_{t\in G} \hat{\phi}_t f = \infty] \supseteq \mathcal{C}\,.$$

If $A \in \mathcal{S}_+ := \{B \in \mathcal{S} :\ \mu(B) > 0\}$ and

$$\sum_{t\in G} \hat{\phi}_t f < \infty$$

on A, then there exists $B \in \mathcal{S}_+\,,\ B \subseteq A$, such that

$$\int_B (\sum_{t\in G} \hat{\phi}_t f)\, d\mu < \infty.$$

It follows that

$$\int_S f.(\sum_{t\in G} I_B \circ \phi_t)\, d\mu < \infty,$$

whence, since $f > 0$ a.e.,

$$\sum_{t\in G} I_B \circ \phi_t < \infty \text{ a.e.}$$

and by Proposition 2.2.2, $B \subseteq \mathcal{D}$. This proves that

$$[\sum_{t\in G} \hat{\phi}_t f = \infty] \supseteq \mathcal{C}\,.$$

Conversely, if $W \in \mathcal{S}_+$ is a wandering set, and $f \in L^1(\mu),\ f > 0$, then

$$\sum_{t \in G} \hat{\phi}_t f < \infty \ \text{ a.e. on } W,$$

since

$$\int_W (\sum_{t \in G} \hat{\phi}_t f)\, d\mu = \int_S f.(\sum_{t \in G} I_W \circ \phi_t)\, d\mu \leq \|f\| < \infty.$$

Thus,

$$[\sum_{t \in G} \hat{\phi}_t f < \infty] \supseteq \mathcal{D}.$$

□

The following is an immediate corollary, particularly suitable for our purposes.

Corollary 2.2.4. *If G is a countable group and* $\{\phi_t\}$ *is a nonsingular action of G then*

$$[\sum_{t \in G} \frac{d\mu \circ \phi_t}{d\mu} f \circ \phi_t = \infty] = C \ \text{ for all } f \in L^1(\mu),\ f > 0.$$

Proof. This follows from Proposition 2.2.3 and the fact that

$$\hat{\phi}_t f = \frac{d\mu \circ \phi_t^{-1}}{d\mu} f \circ \phi_t^{-1} = \frac{d\mu \circ \phi_{t^{-1}}}{d\mu} f \circ \phi_{t^{-1}}\ .$$

□

Note that, as mentioned earlier, the equalities of sets in Proposition 2.2.3 and Corollary 2.2.4 above hold up to sets of μ-measure zero.

2.3 Stationary Symmetric Stable Random Fields

Suppose $\mathbf{X} = \{X_t\}_{t \in \mathbb{Z}^d}$ is a $S\alpha S$ random field, $0 < \alpha < 2$. We know from Theorem 13.1.2 of Samorodnitsky and Taqqu (1994) that it has an integral representation of the from

$$X_t \overset{d}{=} \int_S f_t(s) M(ds), \quad t \in \mathbb{Z}^d, \tag{2.3.1}$$

where M is a $S\alpha S$ random measure on some standard Borel space $(S, \mathcal{S})$ with σ-finite control measure μ and $f_t \in L^\alpha(S,\mu)$ for all $t \in \mathbb{Z}^d$. Note that f_t's are deterministic functions and hence all the randomness of $\mathbf{X}$ is hidden in the random measure M, and the inter-dependence of the X_t's is captured in $\{f_t\}$. The representation (2.3.1) is called an integral representation of $\{X_t\}$. Without loss of generality, we can also assume that the family $\{f_t\}$ satisfies the full support assumption

$$\text{Support}\Big(f_t,\ t \in \mathbb{Z}^d\Big) = S\ , \tag{2.3.2}$$

because, if that is not the case, we can replace S by $S_0 = \text{Support}(f_t,\ t \in \mathbb{Z}^d)$ in (2.3.1).

If, further, $\{X_t\}$ is stationary, then the fact that the action of the group $\mathbb{Z}^d$ on $\{X_t\}_{t\in\mathbb{Z}^d}$ by translation of indices preserves the law, and certain rigidities of spaces L^α, $\alpha < 2$ guarantee existence of integral representations of a special form. We first introduce the following

Definition 2.3.1. *An integral representation* $\{f_t\} \subseteq L^\alpha(S,\mathcal{S},\mu)$ *of a* $S\alpha S$ *random field is said to be minimal if* (2.3.2) *holds, and for every* $B \in \mathcal{S}$ *there exists a* $C \in \sigma\{f_t/f_\tau : t,\tau \in \mathbb{Z}^d\}$ *such that* $\mu(B\Delta C) = 0$.

Existence of minimal representations was proved in Hardin (1982). In general, it is rather difficult to verify whether a given representation is minimal. It has been established in Rosiński (1995) for $d = 1$ and Rosiński (2000) for a general d that every minimal representation of a stationary $S\alpha S$ random field is of the form

$$f_t(s) \;=\; c_t(s)\left(\frac{d\mu\circ\phi_t}{d\mu}(s)\right)^{1/\alpha} f\circ\phi_t(s),\quad t \in \mathbb{Z}^d\,, \tag{2.3.3}$$

where $f \in L^\alpha(S,\mu)$, $\{\phi_t\}_{t\in\mathbb{Z}^d}$ is a nonsingular $\mathbb{Z}^d$-action on (S,μ) and $\{c_t\}_{t\in\mathbb{Z}^d}$ is a measurable cocycle for $\{\phi_t\}$ taking values in $\{-1,+1\}$, i.e., each c_t is a measurable map $c_t : S \to \{-1,+1\}$ such that for all $u, v \in \mathbb{Z}^d$

$$c_{u+v}(s) = c_v(s)c_u(\phi_v(s)) \ \text{ for } \mu\text{-a.a. } s \in S.$$

Conversely, if $\{f_t\}$ is of the form (2.3.3) then $\{X_t\}$ defined by (2.3.1) is a stationary $S\alpha S$ random field.

Although the integral representation of a $S\alpha S$ random field is not completely unique, the following rigidity result is available in Remark 2.5 of Rosiński (1995).

Proposition 2.3.1. *Let* $\{f_t^{(i)}\}_{t\in\mathbb{Z}^d} \subseteq L^\alpha(S_i, \mathcal{S}_i, \mu_i)$, $i = 1, 2$ *be two integral representations of a* $S\alpha S$ *random field* $\{X_t\}$ *such that* $\{f_t^{(1)}\}$ *is minimal and* $\{f_t^{(2)}\}$ *satisfies* $\bigcup_{t\in\mathbb{Z}^d} Support(f_t^{(2)}) = S_2$. *Then there exists measurable functions* $\Phi : S_2 \to S_1$ *and* $h : S_2 \to \mathbb{R} - \{0\}$ *such that, for each* $t \in \mathbb{Z}^d$,

$$f_t^{(2)}(s) = h(s)\, f_t^{(1)}(\Phi(s)) \;\; \text{for } \mu_2\text{-a.a. } s \in S_2\,, \tag{2.3.4}$$

and

$$\mu_1(A) = \int_{\Phi^{-1}(A)} |h(s)|^\alpha \mu_2(ds) \quad \text{for all } A \in \mathcal{S}_1\,. \tag{2.3.5}$$

Remark 2.3.2. In the above proposition, (2.3.4) holds even if we drop the minimality of $\{f_t^{(1)}\}$ (see Theorem 1.1 of Rosiński (1995)).

We will say that a stationary $S\alpha S$ random field $\{X_t\}_{t\in\mathbb{Z}^d}$ is generated by a nonsingular $\mathbb{Z}^d$-action $\{\phi_t\}$ on (S, μ) if it has a integral representation of the form (2.3.3) satisfying (2.3.2). With this terminology, we have the following extension of Theorem 4.1 in Rosiński (1995) to random fields.

Proposition 2.3.3. *Suppose* $\{X_t\}_{t\in\mathbb{Z}^d}$ *is a stationary* $S\alpha S$ *random field generated by a nonsingular* $\mathbb{Z}^d$*-action* $\{\phi_t\}$ *on* (S, μ) *and* $\{f_t\}$ *is given by* (2.3.3). *Also let* $\mathcal{C}$ *and* $\mathcal{D}$ *be the conservative and dissipative parts of* $\{\phi_t\}$. *Then we have*

$$\begin{aligned} \mathcal{C} &= \{s \in S : \sum_{t\in\mathbb{Z}^d} |f_t(s)|^\alpha = \infty\} \;\; \text{mod}\, \mu, \text{ and} \\ \mathcal{D} &= \{s \in S : \sum_{t\in\mathbb{Z}^d} |f_t(s)|^\alpha < \infty\} \;\; \text{mod}\, \mu\,. \end{aligned}$$

In particular, if a stationary $S\alpha S$ random field $\{X_t\}_{t\in\mathbb{Z}^d}$ is generated by a conservative (dissipative, resp.) $\mathbb{Z}^d$-action, then in any other integral representation of $\{X_t\}$ of the form (2.3.3) *satisfying* (2.3.2)*, the $\mathbb{Z}^d$-action must be conservative (dissipative, resp.). Hence the classes of stationary $S\alpha S$ random fields generated by conservative and dissipative actions are disjoint.*

Proof. Define g as

$$g(s) = \sum_{u\in\mathbb{Z}^d} \alpha_u \frac{d\mu\circ\phi_u}{d\mu}(s)|f\circ\phi_u(s)|^\alpha \,,$$

where $\alpha_u > 0$ for all $u \in \mathbb{Z}^d$ and $\sum_{u\in\mathbb{Z}^d} \alpha_u = 1$. Clearly $g \in L^1$ and by (2.3.2), $g > 0$ a.e. μ. Since

$$\sum_{t\in\mathbb{Z}^d} \frac{d\mu\circ\phi_t}{d\mu}(s) g\circ\phi_t(s) = \sum_{t\in\mathbb{Z}^d} \frac{d\mu\circ\phi_t}{d\mu}(s)|f\circ\phi_t(s)|^\alpha = \sum_{t\in\mathbb{Z}^d} |f_t(s)|^\alpha$$

we can use Corollary 2.2.4 to establish the first part of the proposition. For the second part, let $\{\psi_t\}$ be a $\mathbb{Z}^d$-action on (Y,ν) which also generates $\{X_t\}_{t\in\mathbb{Z}^d}$. This means

$$g_t = u_t\left(\frac{d\nu\circ\psi_t}{d\nu}\right)^{1/\alpha} g\circ\psi_t\,, \quad t\in\mathbb{Z}^d$$

is another representation of $\{X_t\}$ satisfying the full support condition (2.3.2), where $g \in L^\alpha(Y,\nu)$, and $\{u_t\}_{t\in\mathbb{Z}^d}$ is a cocycle for $\{\psi_t\}$. We have to show $\{\psi_t\}$ is conservative as well.

Since $\{\phi_t\}$ is conservative, by the first part of this proposition we have $\mu(S - C_0) = 0$ where

$$C_0 := \{s\in S : \sum_{t\in\mathbb{Z}^d} |f_t(s)|^\alpha = \infty\}\,.$$

Since C_0 is $\{\phi_t\}$-invariant, we can restrict $\{f_t\}_{t\in\mathbb{Z}^d}$ to C_0. Call this restriction $\{f_t^0\}_{t\in\mathbb{Z}^d}$. By Remark 2.3.2 there exists measurable functions $\Phi : Y \to C_0$ and $h : Y \to \mathbb{R} - \{0\}$ such that for each $t \in \mathbb{Z}^d$,

$$g_t(y) = h(y)\, f_t^0(\Phi(y)) \;\text{ for } \nu\text{-a.a } y\,.$$

Since $\Phi(y) \in C_0$, we obtain for ν-a.a. y

$$\sum_{t \in \mathbb{Z}^d} |g_t(y)|^\alpha = |h(y)|^\alpha \sum_{t \in \mathbb{Z}^d} \left| f_t(\Phi(y)) \right|^\alpha = \infty \, .$$

Hence $\{\psi_t\}$ is a conservative $\mathbb{Z}^d$-action by another application of the first part of this result. A proof in the case when $\{X_t\}$ is generated by a dissipative $\mathbb{Z}^d$-action is similar.

$\square$

As in the one-dimensional case, it follows that the test described in the previous proposition can be applied to any full support integral representation of the process, not necessarily that of a specific form.

Corollary 2.3.4. *The stationary $S\alpha S$ random field $\{X_t\}_{t \in \mathbb{Z}^d}$ is generated by a conservative (dissipative, resp.) $\mathbb{Z}^d$-action if and only if for any (equivalently, some) integral representation* (2.3.1) *of $\{X_t\}$ satisfying* (2.3.2)*, the sum*

$$\sum_{t \in \mathbb{Z}^d} |f_t(s)|^\alpha$$

is infinite (finite, resp) μ-a.e. .

Proof. Fix a minimal representation $\{f_t^{(1)}\} \subseteq L^\alpha(S_1, \mu_1)$ of $\{X_t\}$ and apply Proposition 2.3.1 with $\{f_t^{(2)}\} = \{f_t\}$. Then, by (2.3.4) we have

$$F = \Phi^{-1}(F_1) \quad mod \; \mu \, , \tag{2.3.6}$$

where

$$F = \{s \in S : \sum_{t \in \mathbb{Z}^d} |f_t(s)|^\alpha < \infty\}, \text{ and}$$
$$F_1 = \{s_1 \in S_1 : \sum_{t \in \mathbb{Z}^d} |f_t^{(1)}(s_1)|^\alpha < \infty\} \, .$$

Hence, by (2.3.5) and (2.3.6),

$$
\begin{aligned}
\sum_{t\in\mathbb{Z}^d} |f_t|^\alpha = \infty \quad \mu\text{-a.e.} \quad &\Longleftrightarrow \quad \mu(F) = 0 \\
&\Longleftrightarrow \quad \int_F |h(s)|^\alpha \mu(ds) = 0 \\
&\Longleftrightarrow \quad \mu_1(F_1) = 0 \\
&\Longleftrightarrow \quad \sum_{t\in\mathbb{Z}^d} \left|f_t^{(1)}\right|^\alpha = \infty \quad \mu\text{-a.e.}\,.
\end{aligned}
$$

Similarly we can show

$$
\sum_{t\in\mathbb{Z}^d} |f_t|^\alpha < \infty \quad \mu\text{-a.e.} \iff \sum_{t\in\mathbb{Z}^d} \left|f_t^{(1)}\right|^\alpha < \infty \quad \mu\text{-a.e.}\,,
$$

and since $\{f_t^{(1)}\}$ is of the form (2.3.3) and satisfies (2.3.2), this corollary follows from Proposition 2.3.3. □

Proposition 2.3.3 also enables us to extend the connection between the structure of stationary stable processes and ergodic theory of nonsingular actions (given in Rosiński (1995)) to the case of stationary stable random fields. A decomposition of a stable random field into three independent parts is available in Rosiński (2000). A connection with the conservative-dissipative decomposition is still missing in the case of random fields. Here we provide the missing link. A stable random field **X** is called a mixed moving average if it can be represented in the form

$$
\mathbf{X} \stackrel{d}{=} \left\{ \int_{W\times\mathbb{Z}^d} f(v, t+s)\, M(dv, ds) \right\}_{t\in\mathbb{Z}^d}, \tag{2.3.7}
$$

where $f \in L^\alpha(W\times\mathbb{Z}^d, \nu\otimes l)$, l is the counting measure on $\mathbb{Z}^d$, ν is a σ-finite measure on a standard Borel space $(W, \mathcal{W})$, and the control measure μ of M equals $\nu\otimes l$ (see Surgailis et al. (1993) and Rosiński (2000)). The following result gives another equivalent characterization of stationary $S\alpha S$ random fields generated by dissipative actions.

Theorem 2.3.5. *Suppose $\{X_t\}_{t\in\mathbb{Z}^d}$ is a stationary $S\alpha S$ random field. Then, the following are equivalent:*

1. *$\{X_t\}$ is generated by a dissipative $\mathbb{Z}^d$-action.*

2. *For any integral representation $\{f_t\}$ of $\{X_t\}$, we have*

$$\sum_{t\in\mathbb{Z}^d} |f_t(s)|^\alpha < \infty \text{ for } \mu\text{-a.a. } s.$$

3. *$\{X_t\}$ is a mixed moving average.*

Proof. 1 and 2 are equivalent by Corollary 2.3.4, and 2 and 3 are equivalent by Theorem 2.1 of Rosiński (2000). □

Theorem 2.3.5 allows us to describe the decomposition of a stationary $S\alpha S$ random field given in Theorem 3.7 of Rosiński (2000) in terms of the ergodic-theoretical properties of nonsingular $\mathbb{Z}^d$-actions generating the field. The statement of the following corollary is an immediate extension of the one-dimensional decomposition in Theorem 4.3 in Rosiński (1995) to random fields.

Corollary 2.3.6. *A stationary $S\alpha S$ random field $\mathbf{X}$ has a unique in law decomposition*

$$X_t \overset{d}{=} X_t^{\mathcal{C}} + X_t^{\mathcal{D}}, \tag{2.3.8}$$

where $\mathbf{X}^{\mathcal{C}}$ and $\mathbf{X}^{\mathcal{D}}$ are two independent stationary $S\alpha S$ random fields such that $\mathbf{X}^{\mathcal{D}}$ is a mixed moving average, and $\mathbf{X}^{\mathcal{C}}$ is generated by a conservative action.

As in the one-dimensional case, it is possible to think of stable random fields generated by conservative actions as having longer memory than those generated by dissipative actions, simply because a conservative action "keeps coming back", and so the same values of the random measure M contribute to observations X_t far separated in t. From this point of view, the $\mathbb{Z}^d$-action $\{\phi_t\}$ is a parameter (though highly infinite-dimensional) of the stationary $S\alpha S$ random field $\{X_t\}$ that determines, among others, the length of its memory.

2.4 Random Fields Indexed by Countable Groups

As mentioned before, all of the structure results of Section 2.3 extend immediately to G-stationary random fields for countable abelian groups G more general than $\mathbb{Z}^d$. The only place where an additional argument is needed is the equivalence of parts 2 and 3 in Theorem 2.3.5, with a G-mixed moving average defined in parallel to (2.3.7). This equivalence needs an extension of Theorem 2.1 in Rosiński (2000) to general countable abelian groups. In this section we establish this extension following verbatim the original proof.

Let $(G, +)$ be a countable abelian group with identity element 0. Assume, in this section, that $\mathbf{X} = \{X_t\}_{t \in G}$ is a stationary $S\alpha S$ random field indexed by G. As in the $\mathbb{Z}^d$ case, $\mathbf{X}$ is called a mixed moving average if it can be represented in the form

$$\mathbf{X} \stackrel{d}{=} \left\{ \int_{W \times G} g(v, t + s)\, M(dv, ds) \right\}_{t \in G} , \tag{2.4.1}$$

where $g \in L^\alpha(W \times G, \nu \otimes l)$, l is the counting measure on G, ν is a σ-finite measure on a standard Borel space $(W, \mathcal{W})$, and the control measure μ of M equals $\nu \otimes l$. The following result is a generalization of Theorem 2.1 in Rosiński (2000) and characterizes mixed moving averages indexed by countable abelian groups.

Theorem 2.4.1. *Suppose $\{f_t : t \in G\} \subset L^\alpha(S, \mu)$ is an arbitrary integral representation of $\mathbf{X}$. Then $\mathbf{X}$ is a mixed moving average if and only if*

$$\sum_{t \in G} |f_t(s)|^\alpha < \infty \qquad \mu\text{-a.e.}\,. \tag{2.4.2}$$

Proof. First we will prove the necessity of (2.4.2). Suppose that $\mathbf{X}$ has a representation (2.4.1). Without loss of generality, we may assume that

$$\text{Support}\big(f_t,\ t \in G\big) = S\ , \tag{2.4.3}$$

and

$$\sum_{t \in G} |g(w,t)|^\alpha < \infty \quad \text{for all } w \in W\,.$$

By Theorem 1.1 of Rosiński (1995), there exist functions $\Phi : S \to W \times G$, $\Phi = (\Phi_1, \Phi_2)$, and $h : S \to \mathbb{R} - \{0\}$ such that

$$f_t(s) = h(s) g(\Phi_1(s), \Phi_2(s) + t) \qquad l \otimes \mu\text{-a.e.},$$

which yields (2.4.2).

The proof of the sufficiency goes through a series of steps modifying the representation $\{f_t\}_{t \in G}$ until the desired form (2.4.1) is obtained.

Step 1. Let $\{g_t : t \in G\} \subset L^\alpha(W, \nu)$ be a minimal representation of $\mathbf{X}$. Then

$$g^*(w) := \sum_{t \in G} |g_t(w)|^\alpha < \infty \qquad \nu\text{-a.e.}. \tag{2.4.4}$$

Proof of Step 1. Since (2.4.3) holds, by Remark 2.5 in Rosiński (1995) there exist measurable maps $\Psi : S \to W$ and $h : S \to \mathbb{R} - \{0\}$ such that, for each $t \in G$,

$$f_t(s) = h(s) g_t(\Psi(s)) \qquad \mu\text{-a.e.} \tag{2.4.5}$$

and $\mu \circ \Psi^{-1} \sim \nu$. Let $W_\infty := \{w : \sum_{t \in G} |g_t(w)|^\alpha = \infty\}$. From (2.4.2) and (2.4.5) we get $\mu(\Psi^{-1}(W_\infty)) = 0$, which gives $\nu(W_\infty) = 0$ and proves (2.4.4). □

Using same arguments as in Theorem 3.1 in Rosiński (1995), we infer that there exists a nonsingular G-action $\{\phi_t\}$ on (W, ν) and a cocycle $c_t : W \to \{-1, +1\}$ such that

$$g_t = c_t \left(\frac{d\nu \circ \phi_t}{d\nu} \right)^{1/\alpha} g_0 \circ \phi_t, \qquad t \in G. \tag{2.4.6}$$

Step 2. Under the assumptions of Step 1, there exists a $\{\phi_t\}_{t \in G}$-invariant measure λ on W which is equivalent to ν.

Proof of Step 2. In view of (2.4.4), the measure λ given by

$$\lambda(dw) := g^*(w)\nu(dw)$$

is absolutely continuous with respect to ν. Let $W_0 := \{w : g^*(w) = 0\}$. We have

$$0 = \int_{W_0} g^* d\nu = \sum_{t \in G} \int_{W_0} |g_t|^\alpha d\nu .$$

Then $\int_{W_0} |g_t|^\alpha d\nu = 0$ for all $t \in G$ and hence W_0 is disjoint (mod ν) with Support($f_t : t \in G$). From the minimality of $\{g_t\}_{t \in G}$ we get $\nu(W_0) = 0$, showing that λ is equivalent to ν.

To prove that λ is invariant, choose $\tau \in G$ and $A \in \mathcal{W}$. We have

$$\begin{aligned}
\lambda(\phi_\tau(A)) &= \int_W I_A(\phi_{-\tau}(w)) g^*(w)\nu(dw) \\
&= \sum_{t \in G} \int_W I_A(\phi_{-\tau}(w)) \frac{d\nu \circ \phi_t}{d\nu}(w) |g_0(\phi_t(w))|^\alpha \nu(dw) \\
&= \sum_{t \in G} \int_W I_A(w) \frac{d\nu \circ \phi_t}{d\nu}(\phi_\tau(w)) |g_0(\phi_{t+\tau}(w))|^\alpha (\nu \circ \phi_\tau(dw)) \\
&= \int_W I_A(w) \left(\sum_{t \in G} \frac{d\nu \circ \phi_{t+\tau}}{d\nu}(w) |g_0(\phi_{t+\tau}(w)|^\alpha \right) \nu(dw) = \lambda(A) .
\end{aligned}$$

This completes the proof of Step 2.

Step 3. Define

$$h_t(w) := c_t(w) h(\phi_i(w)) ,$$

where $h := (g^*)^{-1/\alpha} g_0$. Then $\{h_t : t \in G\} \subset L^\alpha(W, \lambda)$ is a representation of $\mathbf{X}$ such that, for λ-a.a. $w \in W$,

$$\sum_{t \in G} |h(\phi_t(w))|^\alpha = 1 . \tag{2.4.7}$$

Proof of Step 3. From the equality

$$1 = \frac{d\lambda \circ \phi_t}{d\lambda} = \frac{d\nu \circ \phi_t}{d\nu} \frac{g^* \circ \phi_t}{g^*}$$

we get

$$g_t = c_t \left(\frac{g^*}{g^* \circ \phi_t} \right)^{1/\alpha} g_0 \circ \phi_t = (g^*)^{1/\alpha} h_t$$

or

$$h_t = (g^*)^{-1/\alpha} g_t \,,$$

proving that $\{h_t\}_{t \in G} \subset L^\alpha(W, \lambda)$ is a representation of $\mathbf{X}$. Since the last equality holds mod λ, for each $t \in G$, (2.4.7) follows.

Notice now that the set $\{w : \sum_{t \in G} |h(\phi_t(w))|^\alpha = 1\}$ is $\{\phi_t\}$-invariant. Therefore, removing from W the complement of this set, which is of measure zero by (2.4.7), does not affect the representation $\{h_t\}_{t \in G}$. Hence we may and do assume that (2.4.7) holds for all $w \in W$.

Step 4. There exists a sequence of $\{\phi_t\}$-invariant real-valued Borel functions on W which separate the orbits of $\{\phi_t\}_{t \in G}$.

Proof of Step 4. We will now employ some topological arguments. By Theorem 8.7 of Varadarajan (1970), W can be considered as a Borel subset of a compact metric space $\widetilde{W}$ on which the action $\{\phi_t\}_{t \in G}$ is defined, W is $\{\phi_t\}$-invariant, and for all $t \in G$, $w \mapsto \phi_t(w)$ is a continuous map on $\widetilde{W}$.

Let $\{A_n\}$ be the sequence of finite unions of finite intersections of sets from a countable topological basis of W. Let

$$A_{nm} := A_n \cap \{w \in W : |h(w)| > m^{-1}\} \,.$$

Since $\int_W |h|^\alpha d\lambda < \infty$, $\lambda(A_{nm}) < \infty$ for every $n, m \geq 1$. Define

$$u_{nm}(w) := \sum_{t \in G} I_{A_{nm}}(\phi_t(w)) \,.$$

Notice that

$$u_{nm}(w) \le \sum_{t \in G} m^{\alpha} |h(\phi_t(w))|^{\alpha} = m^{\alpha} < \infty \,,$$

for every $w \in W$, $n, m \le 1$ and clearly u_{nm} is $\{\phi_t\}$-invariant. We will show that $\{u_{nm}\}_{n,m \ge 1}$ separate the orbits of $\{\phi_t\}_{t \in G}$.

Suppose that w_1 and w_2 live on different orbits. We first claim that for some $n, m \ge 1$,

$$u_{nm}(w_1) \ge 1 \,. \tag{2.4.8}$$

Indeed, from (2.4.7) (which now holds for all $w \in W$) we infer that there exist $m \ge 1$ and $t_0 \in G$ such that $|h(\phi_{t_0}(w_1))| > m^{-1}$. Furthermore, there exists $n \ge 1$ such that $\phi_{t_0}(w_1) \in A_n$. Hence, $\phi_{t_0}(w_1) \in A_{nm}$, which proves (2.4.8). Now we will show that w_1 and w_2 can be separated.

Let n, m be as in (2.4.8). If $u_{nm}(w_1) \ne u_{nm}(w_2)$, then there is nothing to prove. Thus we assume

$$u_{nm}(w_2) = u_{nm}(w_1) = M \ge 1.$$

There exists a finite set $K \subset G$ such that

$$\sum_{t \in K} I_{A_{nm}}(\phi_t(w_i)) > \frac{M}{2} \,, \quad i = 1, 2. \tag{2.4.9}$$

Consider the finite subsets of W given by

$$\Gamma_i := \{\phi_t(w_i) : \ t \in K\} \,, \quad i = 1, 2.$$

Since w_1 and w_2 live on different orbits, $\Gamma_1 \cap \Gamma_2 = \phi$. Hence, there is an $n_1 \ge 1$ such that

$$A_{n_1} \supset \Gamma_1 \ \text{ and } \ A_{n_1} \cap \Gamma_2 = \phi \,.$$

By the definition of the sequence $\{A_k\}$, $A_n \cap A_{n_1} = A_{n'}$, for some $n' \ge 1$. Now consider $u_{n'm}$. In view of (2.4.9) we get

$$u_{n'm}(w_1) \ge \sum_{t \in G} I_{\Gamma_1 \cap A_{nm}}(\phi_t(w_1)) \ge \sum_{t \in K} I_{A_{nm}}(\phi_t(w_1)) > \frac{M}{2}$$

and

$$u_{n'm}(w_2) = \sum_{t\in K^c} I_{A_{n'm}}(\phi_t(w_2)) < \frac{M}{2}\,.$$

Hence $u_{n'm}(w_2) < u_{n'm}(w_1)$, which ends the proof of Step 4.

By Step 4 and von Neumann's cross-section lemma (see Corollary 8.2 in Varadarajan (1970), there exists a Borel set $W_0 \subset W$ which intersects each orbit of $\{\phi_t\}_{t\in G}$ at exactly one point. (To be exact, this step may require a reduction of W to some $\{\phi_t\}_{t\in G}$-invariant subset, say, $\widetilde{W}$, such that $\lambda(W - \widetilde{W}) = 0$.)

Step 5. Let W_0 be as above. The map $\Phi : W_0 \times G \to W$, given by $\Phi(w_0, t) = \phi_t(w_0)$, is a Borel isomorphism. The measure $\lambda \circ \Phi$, induced on $W_0 \times G$ by the inverse map Φ^{-1} from W, is the product measure of certain measure λ_0 on W_0 and the counting measure on G.

Proof of Step 5. First we will show that Φ is one-to-one. Suppose that $\Phi(w_1, t_1) = \Phi(w_2, t_2)$. Then $w_1 = w_2 = w_0$ from the definition of W_0. Thus $\phi_{t_0}(w_0) = w_0$ where $t_0 = t_1 - t_2$. Hence $h(\phi_{nt_0}(w_0)) = h(w_0)$, for every integer n, which implies $t_0 = 0$ since (2.4.7) holds for all $w \in W$. Hence Φ is one-to-one and clearly onto. Since Φ is Borel measurable, its inverse is measurable by Kuratowski's Theorem.

Let $\Phi^{-1}(w) = (\pi(w), \tau(w))$ and consider $Q := \lambda \circ \Phi$. We have

$$Q(A \times B) = \lambda(\{w : \pi(w) \in A, \tau(w) \in B\}), \quad A \in \mathcal{W},\ B \subseteq G\,.$$

Since λ is $\{\phi_t\}$-invariant by Step 2, we get

$$\begin{aligned} Q(A \times (B + t)) &= \lambda(\{w : \pi(\phi_t(w)) \in A, \tau(\phi_t(w)) \in B + t\}) \\ &= \lambda(\{w : \pi(w) \in A, \tau(w) + t \in B + t\}) = Q(A \times B)\,. \end{aligned}$$

Hence $Q(A \times \cdot)$ is proportional to the counting measure on G, and consequently, $Q(A \times B) = \lambda_0(A)|B|$, for some measure λ_0 on W_0.

The next step ends the proof of the theorem.

Step 6. By Theorem B.9 in the Appendix of Zimmer (1984) it follows that we can choose $\{c_t\}_{t \in G}$ in (2.4.6) in such a way that for all $(u, v, w) \in G \times G \times W$

$$c_{u+v}(w) = c_v(w)c_u(\phi_v(w)) .$$

Define

$$k(w_0, s) := c_s(w_0)h(\phi_s(w_0)) \quad (w_0, s) \in W_0 \times G .$$

Then $k_t(w_0, s) := k(w_0, t + s)$ is a representation of $\mathbf{X}$ in $L^\alpha(W_0 \times G, \lambda_0 \otimes l)$.

Proof of Step 6. For every $a_1, a_2, \ldots, a_n \in \mathbb{R}$ and $t_1, t_2, \ldots, t_n \in G$ we have

$$\begin{aligned}
&\int_{W_0 \times G} \Big| \sum a_j k_{t_j}(w_0, s) \Big|^\alpha \lambda_0(dw_0)\, l(ds) \\
&\quad = \int_{W_0 \times G} \Big| \sum a_j c_{t_j}(\Phi(w_0, s)) h(\phi_{t_j}(\Phi(w_0, s))) \Big|^\alpha \lambda_0(dw_0)\, l(ds) \\
&\quad = \int_W \Big| \sum a_j h_{t_j}(w) \Big|^\alpha \lambda(dw)
\end{aligned}$$

which ends the proof of Theorem 2.4.1. □

Remark 2.4.2. Theorem 2.3.5 (and hence Corollary 2.3.6) holds in the countable abelian group case as well. Since all the other structure results extend easily, the equivalence of 1 and 2 can be established in the same fashion as in the $\mathbb{Z}^d$ case. Equivalence of 2 and 3 follows from Theorem 2.4.1.

2.5 The Sequence $\mathbf{b_n}$

The length of memory of stable random fields is manifested, in particular, in the rate of growth of its extreme values. If X_t is generated by a conservative action, the extreme values tend to grow at a slower rate because longer memory prevents erratic changes in X_t even when t becomes "large". This has been formalized in Samorodnitsky (2004a) for $d = 1$, and it turns out to be the case for stable random fields as well.

For a stationary $S\alpha S$ random field $\{X_t\}_{t\in\mathbb{Z}^d}$, we will study the partial maxima sequence

$$M_n \;:=\; \max_{0\le t\le(n-1)\mathbf{1}} |X_t|, \quad n = 0, 1, 2, \ldots \tag{2.5.1}$$

where $u = (u^{(1)}, u^{(2)}, \ldots, u^{(d)}) \le v = (v^{(1)}, v^{(2)}, \ldots, v^{(d)})$ means $u^{(i)} \le v^{(i)}$ for all $i = 1, 2, \ldots, d$ and $\mathbf{1} = (1, 1, \ldots, 1)$. As in the one-dimensional case, the asymptotic behavior of the maximum functional M_n is related to the deterministic sequence

$$b_n \;:=\; \left(\int_S \max_{0\le t\le(n-1)\mathbf{1}} |f_t(s)|^\alpha \mu(ds)\right)^{1/\alpha}, \quad n = 0, 1, 2, \ldots\,. \tag{2.5.2}$$

In fact, to a certain extent b_n controls "the size" of M_n even without the assumption of stationarity of the random field. Indeed, for any $0 < p < \alpha$, (see Theorem 2.1 of Marcus (1984)) there are constants $c_{\alpha,p}, C_{\alpha,p} \in (0,\infty)$ such that, for $1 < \alpha < 2$,

$$c_{\alpha,p} \le \;\frac{1}{b_n}(EM_n^p)^{1/p}\; \le C_{\alpha,p}\,(\log n)^{1/\alpha'}, \tag{2.5.3}$$

where α' is such that $1/\alpha + 1/\alpha' = 1$, while for $\alpha = 1$,

$$c_{1,p} \le \;\frac{1}{b_n}(EM_n^p)^{1/p}\; \le C_{1,p}\,L_n\,, \tag{2.5.4}$$

where $L_n := \max(1, \log d + \log\log n)$, and for $0 < \alpha < 1$,

$$c_{1,p} \le \;\frac{1}{b_n}(EM_n^p)^{1/p}\; \le C_{1,p}\,. \tag{2.5.5}$$

Note that b_n is completely determined by the process, and does not depend on a particular integral representation (see Corollary 4.4.6 of Samorodnitsky and Taqqu (1994)). We are interested in the features of this sequence that are related to the decomposition of a stable random field in Corollary 2.3.6. The next result shows that the sequence b_n grows at a slower rate for random fields generated by a conservative action than for random fields generated by a dissipative action.

Proposition 2.5.1. *Let* $\{f_t\}$ *be given by* (2.3.3). *Assume that* (2.3.2) *holds.*

1. *If the action* $\{\phi_t\}$ *is conservative then:*

$$n^{-d/\alpha} b_n \quad \to \quad 0 \quad \text{as } n \to \infty. \tag{2.5.6}$$

2. *If the action* $\{\phi_t\}$ *is dissipative, and the random field is given in the mixed moving average form* (2.3.7), *then:*

$$\lim_{n\to\infty} n^{-d/\alpha} b_n \quad = \quad \left(\int_W (g(v))^\alpha \nu(dv)\right)^{1/\alpha} \in (0, \infty), \tag{2.5.7}$$

where

$$g(v) = \sup_{s\in\mathbb{Z}^d} |f(v, s)| \quad \text{for } v \in W. \tag{2.5.8}$$

Proof. 1. Firstly we observe that, without loss of generality, we can assume that μ is a probability measure. This is because if ν is a probability measure equivalent to the σ-finite measure μ then instead of (2.3.1) we will use

$$X_t \stackrel{d}{=} \int_S h_t(s) N(ds)$$

where

$$h_t(s) = c_t(s)\left(\frac{d\nu \circ \phi_t}{d\nu}(s)\right)^{1/\alpha} h \circ \phi_t(s), \quad t \in \mathbb{Z}^d$$

where $h = f(\frac{d\mu}{d\nu})^{1/\alpha} \in L^\alpha(S, \nu)$ and N is a $S\alpha S$ random measure on S with control measure ν.

Since $\{b_n\}$ is an increasing sequence, it is enough to show (2.5.6) along the odd subsequence. By stationarity of $\{X_t\}$, we need to check that

$$a_n := \frac{1}{(2n+1)^d} \int_S \max_{t \in J_n} |f_t(s)|^\alpha \mu(ds) \to 0 \,,$$

where $J_n := \{(i_1, i_2, \ldots, i_d) : -n \leq i_1, i_2, \ldots, i_d \leq n\}$. Let $g = |f|^\alpha$. Then $\|g\| := \int_S g(s)\mu(ds) < \infty$, and we have for $0 < \epsilon < 1$

$$\begin{aligned} a_n &= \frac{1}{(2n+1)^d} \int_S \max_{t \in J_n} \hat{\phi}_t g(s)\mu(ds) \\ &\leq \frac{1}{(2n+1)^d} \Big(\int_S \max_{t \in J_n} \Big[\hat{\phi}_t g(s) I\Big(\hat{\phi}_t g(s) \leq \epsilon \sum_{u \in J_n} \hat{\phi}_u g(s) \Big) \Big] \mu(ds) \\ &\qquad + \int_S \max_{t \in J_n} \Big[\hat{\phi}_t g(s) I\Big(\hat{\phi}_t g(s) > \epsilon \sum_{u \in J_n} \hat{\phi}_u g(s) \Big) \Big] \mu(ds) \Big) \\ &=: a_n^{(1)} + a_n^{(2)} \,. \end{aligned}$$

Clearly,

$$a_n^{(1)} \leq \frac{\epsilon}{(2n+1)^d} \sum_{u \in J_n} \int_S \hat{\phi}_u g(s)\mu(ds) = \epsilon \|g\| \,, \tag{2.5.9}$$

and

$$a_n^{(2)} \leq \frac{1}{(2n+1)^d} \sum_{t \in J_n} \int_S \hat{\phi}_t g(s) I_{A_{t,n}}(s)\mu(ds) \,, \tag{2.5.10}$$

where $A_{t,n} = \{s : \hat{\phi}_t g(s) > \epsilon \sum_{u \in J_n} \hat{\phi}_u g(s)\}$, $n \geq 1$, $t \in J_n$. Notice that for all $n \geq 1$, and for all $t \in J_n$,

$$\int_S \hat{\phi}_t g(s) I_{A_{t,n}}(s)\mu(ds) = \int_S g(s) I_{\phi_t^{-1}(A_{t,n})}(s)\mu(ds) \,. \tag{2.5.11}$$

The following is the most important step of this proof: if we define

$$U_n := \{(t_1, t_2, \ldots, t_d) : -n + [\sqrt{n}] \leq t_1, t_2, \ldots, t_d \leq n - [\sqrt{n}]\}$$

then we have

$$\lim_{n \to \infty} \max_{t \in U_n} \mu(\phi_t^{-1}(A_{t,n})) = 0 \,. \tag{2.5.12}$$

To prove (2.5.12) observe that for all $t \in U_n$

$$
\begin{aligned}
&\phi_t^{-1}(A_{t,n}) \\
&\quad = \Big\{\phi_{-t}(s) : \ g \circ \phi_{-t}(s) \frac{d\mu \circ \phi_{-t}}{d\mu}(s) > \epsilon \sum_{u \in J_n} g \circ \phi_{-u}(s) \frac{d\mu \circ \phi_{-u}}{d\mu}(s)\Big\} \\
&\quad = \Big\{s : \ g(s) > \epsilon \sum_{u \in J_n} g \circ \phi_{u+t}(s) \frac{d\mu \circ \phi_{u+t}}{d\mu}(s)\Big\} \\
&\quad \subseteq \Big\{s : \ g(s) > \epsilon \sum_{\tau \in J_{[\sqrt{n}]}} g \circ \phi_{\tau}(s) \frac{d\mu \circ \phi_{\tau}}{d\mu}(s)\Big\}.
\end{aligned}
$$

The last inclusion holds because $J_{[\sqrt{n}]} \subseteq t + J_n$. Hence, for any $M > 0$

$$
\begin{aligned}
\max_{t \in U_n} \ \mu(\phi_t^{-1}(A_{t,n})) \ &\leq \ \mu\{s : \ g(s) > \epsilon M\} + \mu\Big(\sum_{t \in J_{[\sqrt{n}]}} g \circ \phi_t \frac{d\mu \circ \phi_t}{d\mu} \leq M\Big) \\
&\leq \ \frac{\|g\|}{\epsilon M} + \mu\Big(\sum_{t \in J_{[\sqrt{n}]}} |f_t|^{\alpha} \leq M\Big).
\end{aligned}
$$

Now (2.5.12) follows by first using Proposition 2.3.3 with a fixed M and then letting $M \to \infty$.

From (2.5.11) and (2.5.12) it follows that

$$
\begin{aligned}
&\frac{1}{(2n+1)^d} \sum_{t \in U_n} \int_S \hat{\phi}_t g(s) I_{A_{t,n}}(s) \mu(ds) \\
&\qquad\qquad = \frac{1}{(2n+1)^d} \sum_{t \in U_n} \int_{\phi_t^{-1}(A_{t,n})} g(s) \mu(ds) \to 0\,. \qquad (2.5.13)
\end{aligned}
$$

If we define $V_n = J_n - U_n$, then

$$
\frac{1}{(2n+1)^d} \sum_{t \in V_n} \int_S \hat{\phi}_t g(s) I_{A_{t,n}}(s) \mu(ds) \leq \frac{1}{(2n+1)^d} \sum_{t \in V_n} \int_S \hat{\phi}_t g(s) \mu(ds) \to 0\,.
$$

Then using (2.5.10) and (2.5.13) we see that $a_n^{(2)} \to 0$ as $n \to \infty$. Therefore we get,

$$
\limsup a_n \leq \limsup a_n^{(1)} + \limsup a_n^{(2)} \leq \epsilon \|g\|\,,
$$

and since $\epsilon > 0$ is arbitrary, the result follows.

2. We start with the case where f has compact support, that is

$$f(v,s)I_{W\times[-m\mathbf{1},m\mathbf{1}]^c}(v,s)\equiv 0 \text{ for some } m=1,2,\ldots$$

where $[u,v]:=\{t\in\mathbb{Z}^d : u\le t\le v\}$. In this case, we have

$$\begin{aligned}
b_n^\alpha &= \sum_{s\in\mathbb{Z}^d}\int_W \max_{0\le t\le(n-1)\mathbf{1}}|f(v,s+t)|^\alpha\nu(dv)\\
&= \sum_{(-m-n+1)\mathbf{1}\le s\le m\mathbf{1}}\int_W \max_{0\le t\le(n-1)\mathbf{1}}|f(v,s+t)|^\alpha\nu(dv)\\
&= \sum_{s\in A_n}\int_W \max_{0\le t\le(n-1)\mathbf{1}}|f(v,s+t)|^\alpha\nu(dv)\\
&\quad+\sum_{s\in B_n}\int_W \max_{0\le t\le(n-1)\mathbf{1}}|f(v,s+t)|^\alpha\nu(dv) =: T_n+R_n
\end{aligned}$$

where

$$\begin{aligned}
A_n &:= [(m-n-1)\mathbf{1},-m\mathbf{1}] \text{ and}\\
B_n &:= [(-m-n+1)\mathbf{1},m\mathbf{1}]-[(m-n-1)\mathbf{1},-m\mathbf{1}].
\end{aligned}$$

Observe that, for $n\ge 2m+1$ we have for each $s\in A_n$,

$$\max_{0\le t\le(n-1)\mathbf{1}}|f(v,s+t)|=g(v)$$

while

$$\max_{0\le t\le(n-1)\mathbf{1}}|f(v,s+t)|\le g(v)$$

for each $s\in B_n$, and so,

$$\begin{aligned}
T_n &= (n-2m)^d\int_W (g(v))^\alpha\nu(dv) \text{ and}\\
R_n &\le [(2m+n)^d-(n-2m)^d]\int_W (g(v))^\alpha\nu(dv).
\end{aligned}$$

Therefore (2.5.7) follows when f has compact support.

In the general case, given $\epsilon>0$, choose a compact supported f_ϵ such that $|f_\epsilon(v,s)|\le|f(v,s)|$ for all v,s and

$$\sum_{s\in\mathbb{Z}^d}\int_W|f(v,s)|^\alpha\nu(dv)-\sum_{s\in\mathbb{Z}^d}\int_W|f_\epsilon(v,s)|^\alpha\nu(dv)\le\epsilon.$$

Let

$$g_\epsilon(v) = \sup_{s\in\mathbb{Z}^d} |f_\epsilon(v,s)|, \quad v \in W.$$

Then

$$\begin{aligned}
0 &\le \int_W (g(v))^\alpha \nu(dv) - \int_W (g_\epsilon(v))^\alpha \nu(dv) \\
&\le \int_W \sup_{s\in\mathbb{Z}^d}(|f(v,s)|^\alpha - |f_\epsilon(v,s)|^\alpha)\nu(dv) \\
&\le \int_W \sum_{s\in\mathbb{Z}^d}(|f(v,s)|^\alpha - |f_\epsilon(v,s)|^\alpha)\nu(dv) \\
&= \sum_{s\in\mathbb{Z}^d}\int_W |f(v,s)|^\alpha \nu(dv) - \sum_{s\in\mathbb{Z}^d}\int_W |f_\epsilon(v,s)|^\alpha \nu(dv) \le \epsilon.
\end{aligned}$$

Therefore,

$$\begin{aligned}
|\frac{1}{n^d}b_n^\alpha &- \int_W (g(v))^\alpha \nu(dv)| \\
&\le \frac{1}{n^d}|\sum_{s\in\mathbb{Z}^d}\int_W \max_{0\le t\le(n-1)\mathbf{1}} |f(v,s+t)|^\alpha \nu(dv) \\
&\qquad - \sum_{s\in\mathbb{Z}^d}\int_W \max_{0\le t\le(n-1)\mathbf{1}} |f_\epsilon(v,s+t)|^\alpha \nu(dv)| \\
&+ |\frac{1}{n^d}\sum_{s\in\mathbb{Z}^d}\int_W \max_{0\le t\le(n-1)\mathbf{1}} |f_\epsilon(v,s+t)|^\alpha \nu(dv) - \int_W (g_\epsilon(v))^\alpha \nu(dv)| \\
&+ |\int_W (g_\epsilon(v))^\alpha \nu(dv) - \int_W (g(v))^\alpha \nu(dv)| =: T_n^{(1)} + T_n^{(2)} + T_n^{(3)}.
\end{aligned}$$

By the above, $T_n^{(3)} \le \epsilon$, and the same argument shows that $T_n^{(1)} \le \epsilon$ as well. Furthermore, by already considered compact support case, $T_n^{(2)} \to 0$ as $n \to \infty$. Hence

$$\limsup_{n\to\infty} |\frac{1}{n^d}b_n^\alpha - \int_W (g(v))^\alpha \nu(dv)| \le 2\epsilon,$$

and since $\epsilon > 0$ is arbitrary, the proof of (2.5.7) is complete. □

Remark 2.5.2. The statement of the first part of the proposition clearly extends to G-stationary random fields for any free abelian group G of rank d, since the same is true for Proposition 2.3.3.

2.6 Maxima of Stable Random Fields

We are now ready to investigate the rate of growth of the sequence $\{M_n\}$ of partial maxima of a stationary symmetric α-stable random field, $0 < \alpha < 2$. We will see that if such a random field has a nonzero component $X^{\mathcal{D}}$ in (2.3.8) generated by a dissipative action, then the partial maxima grow at the rate $n^{d/\alpha}$, while if the random field is generated by a conservative action, then the partial maxima grow at a slower rate. As we will see in the sequel, the actual rate of growth of the sequence $\{M_n\}$ in the conservative case, depends on a number of factors. The dependence on the group theoretical properties of the action is very prominent. We start with the following result, which extends Theorem 4.1 of Samorodnitsky (2004a) to $d > 1$ based on Proposition 2.5.1.

Theorem 2.6.1. *Let* $\mathbf{X} = \{X_t\}_{t\in\mathbb{Z}^d}$ *be a stationary* $S\alpha S$ *random field, with* $0 < \alpha < 2$*, integral representation* (2.3.1)*, and functions* $\{f_t\}$ *given by* (2.3.3).

1. Suppose that $\mathbf{X}$ *is not generated by a conservative action (i.e., the component* $X^{\mathcal{D}}$ *in* (2.3.8) *generated by the dissipative part is nonzero). Then*

$$\frac{1}{n^{d/\alpha}} M_n \Rightarrow C_\alpha^{1/\alpha} K_X Z_\alpha \tag{2.6.1}$$

as $n \to \infty$*, where*

$$K_X = \left(\int_W (g(v))^\alpha \nu(dv) \right)^{1/\alpha}$$

and g *is given by* (2.5.8) *for any representation of* $X^{\mathcal{D}}$ *in the mixed moving average form* (2.3.7)*,* C_α *is the stable tail constant given by*

$$C_\alpha = \left(\int_0^\infty x^{-\alpha} \sin x \, dx \right)^{-1} = \begin{cases} \frac{1-\alpha}{\Gamma(2-\alpha)\cos(\pi\alpha/2)}, & \text{if } \alpha \neq 1, \\ \frac{2}{\pi}, & \text{if } \alpha = 1, \end{cases} \tag{2.6.2}$$

and Z_α *is the standard Frechet-type extreme value random variable with the distribution*

$$P(Z_\alpha \le z) = e^{-z^{-\alpha}}, \quad z > 0. \tag{2.6.3}$$

2. Suppose that $\mathbf{X}$ *is generated by a conservative* $\mathbb{Z}^d$*-action. Then*

$$\frac{1}{n^{d/\alpha}} M_n \xrightarrow{p} 0 \tag{2.6.4}$$

as $n \to \infty$*. Furthermore, with* b_n *given by* (2.5.2)*,*

$$\left\{ \frac{1}{c_n} M_n \right\} \text{ is not tight for any positive sequence } c_n = o(b_n), \tag{2.6.5}$$

while

$$\left\{ \frac{1}{b_n \zeta_n} M_n \right\} \text{ is tight, where } \zeta_n = \begin{cases} 1, & \text{if } 0 < \alpha < 1, \\ L_n, & \text{if } \quad \alpha = 1, \\ (\log n)^{1/\alpha'}, & \text{if } 1 < \alpha < 2, \end{cases} \tag{2.6.6}$$

where $L_n := \max(1, \log d + \log \log n)$*, and for* $\alpha > 1$*,* α' *is such that* $1/\alpha + 1/\alpha' = 1$*. If, for some* $\theta > 0$ *and* $c > 0$*,*

$$b_n \geq c n^{\theta} \quad \text{ for all } n \geq 1, \tag{2.6.7}$$

then (2.6.6) *holds with* $\zeta_n \equiv 1$ *for all* $0 < \alpha < 2$*. Finally, for n=1,2,…, let* η_n *be a probability measure on* $(S, \mathcal{S})$ *with*

$$\frac{d\eta_n}{d\mu}(s) = b_n^{-\alpha} \max_{0 \leq t \leq (n-1)\mathbf{1}} |f_t(s)|^{\alpha}, \qquad s \in S, \tag{2.6.8}$$

and let $U_j^{(n)}$*,* $j = 1, 2$ *be independent* S*-valued random variables with common law* η_n*. Suppose that* (2.6.7) *holds and for any* $\epsilon > 0$*,*

$$P\Bigg(\text{for some } t \in [0, (n-1)\mathbf{1}], \\ \frac{|f_t(U_j^{(n)})|}{\max_{0 \leq u \leq (n-1)\mathbf{1}} |f_u(U_j^{(n)})|} > \epsilon, \ j = 1, 2 \Bigg) \to 0 \tag{2.6.9}$$

as $n \to \infty$*. Then*

$$\frac{1}{b_n} M_n \Rightarrow C_\alpha^{1/\alpha} Z_\alpha \tag{2.6.10}$$

as $n \to \infty$*.*

Remark 2.6.2. An easily verifiable sufficient condition for (2.6.9) is

$$\lim_{n\to\infty} \frac{b_n}{n^{d/2\alpha}} = \infty. \tag{2.6.11}$$

Let r_n denote the probability in the left-hand side of (2.6.9). Clearly,

$$r_n \le \sum_{0\le t\le (n-1)\mathbf{1}} \left(P\Big(\frac{|f_t(U_1^{(n)})|}{\max_{0\le u\le (n-1)\mathbf{1}} |f_u(U_1^{(n)})|} > \epsilon \Big) \right)^2 .$$

Furthermore, for every $t \in [0, (n-1)\mathbf{1}]$,

$$\begin{aligned} &P\Big(\frac{|f_t(U_1^{(n)})|}{\max_{0\le u\le (n-1)\mathbf{1}} |f_u(U_1^{(n)})|} > \epsilon \Big) \\ &= b_n^{-\alpha} \int_S I\Big(\frac{|f_t(s)|}{\max_{0\le u\le (n-1)\mathbf{1}} |f_u(s)|} > \epsilon \Big) \max_{0\le u\le (n-1)\mathbf{1}} |f_t(s)|^\alpha \mu(ds) \\ &\le \epsilon^{-\alpha} b_n^{-\alpha} \int_S |f_t(s)|^\alpha \mu(ds), \end{aligned}$$

and (2.6.9) follows from (2.6.11) since, by the stationarity, the last integral does not depend on t.

Alternatively, (2.6.9) holds if we assume that μ is a finite measure, $\{\phi_t\}$ is measure preserving, the sequence $\{b_n^{-\alpha} \max_{0\le t\le (n-1)\mathbf{1}} |f_t(s)|^\alpha\}$, $t \in \mathbb{Z}^d$ is uniformly integrable with respect to μ and for every $\epsilon > 0$

$$\lim_{n\to\infty} n^{d/2} \mu\{s \in S : |f(s)| > \epsilon b_n\} = 0\,. \tag{2.6.12}$$

Let $\|\mu\|$ denote the total mass of μ. Given a $\delta > 0$, select $M > 0$ such that

$$\int_S I\Big(\max_{0\le t\le (n-1)\mathbf{1}} |f_t(s)|^\alpha > M b_n^\alpha \Big) \max_{0\le t\le (n-1)\mathbf{1}} |f_t(s)|^\alpha \mu(ds) \le \delta b_n^\alpha$$

for all $n \ge 1$. We have with ϵ from (2.6.9),

$$\begin{aligned} r_n \le 4\delta^2 \quad + \quad & b_n^{-2\alpha} \sum_{0\le t\le (n-1)\mathbf{1}} \Big(\int_S \max_{0\le u\le (n-1)\mathbf{1}} |f_u(s)|^\alpha \\ & \qquad \times I\Big(\delta \|\mu\|^{-1} b_n^\alpha \le \max_{0\le u\le (n-1)\mathbf{1}} |f_u(s)|^\alpha \le M b_n^\alpha \Big) \\ & \qquad \times I\Big(\frac{|f_t(s)|}{\max_{0\le u\le (n-1)\mathbf{1}} |f_u(s)|^\alpha} \Big) \mu(ds) \Big)^2 \\ \le 4\delta^2 \quad + \quad & M^2 n^d \Big(\int_S I(|f(s)| > \epsilon \delta^{1/\alpha} \|\mu\|^{-1/\alpha} b_n) \mu(ds) \Big)^2 . \end{aligned}$$

Therefore, using (2.6.12), we obtain,

$$\limsup_{n\to\infty} r_n \leq 4\delta^2,$$

and (2.6.9) follows by letting $\delta \to 0$.

Proof of Theorem 2.6.1. We start by observing that (2.6.6) follows from (2.5.3) - (2.5.5) regardless of the properties of the action. For the rest of the proof, we will use the a series representation of the random vector $(X_t, 0 \leq t \leq (n-1)\mathbf{1})$ of the form

$$X_t \stackrel{d}{=} b_n C_\alpha^{1/\alpha} \sum_{j=1}^{\infty} \varepsilon_j \Gamma_j^{-1/\alpha} \frac{|f_t(U_j^{(n)})|}{\max_{0\leq u\leq(n-1)\mathbf{1}} |f_u(U_j^{(n)})|}, \qquad t \in [0,(n-1)\mathbf{1}] \tag{2.6.13}$$

where C_α is given by (2.6.2), $\varepsilon_1, \varepsilon_2, \ldots$ are i.i.d. Rademacher random variables (symmetric ± 1-valued random variables), $\Gamma_1, \Gamma_2, \ldots$ is a sequence of the arrival times of a unit rate Poisson process on $(0,\infty)$, and $\{U_j^{(n)}\}$ are i.i.d. S-valued random variables with common law given by (2.6.8). All three sequences are independent. This series representation is available in Section 3.10 of Samorodnitsky and Taqqu (1994).

We use the above representation and symmetry to prove (2.6.5). For each n let $\mathcal{G}_n$ be the σ-field generated by ε_1, $\{\Gamma_j\}_{j\geq 1}$ and $\{U_j^{(n)}\}_{j\geq 1}$. Letting

$$Z_n = b_n C_\alpha^{1/\alpha} \max_{0\leq t\leq(n-1)\mathbf{1}} \left| \Gamma_1^{-1/\alpha} \frac{|f_t(U_1^{(n)})|}{\max_{0\leq u\leq(n-1)\mathbf{1}} |f_u(U_1^{(n)})|} \right|$$

and T_0 be the smallest (in lexicographical order) $t \in [0,(n-1)\mathbf{1}]$ over which the maximum is achieved, we see that both Z_n and T_0 are measurable w.r.t. $\mathcal{G}_n$. Further, the symmetry tells us that, for any $x > 0$,

$$P(|X_{T_0}| > x|\mathcal{G}_n) \geq \frac{1}{2} P(Z_n > x|\mathcal{G}_n).$$

Hence, for any $x > 0$,

$$
\begin{aligned}
&P\Big(\frac{1}{c_n} M_n > x\Big) \\
&\quad \geq \frac{1}{2} P\Big(b_n C_\alpha^{1/\alpha} \max_{0 \leq t \leq (n-1)\mathbf{1}} \Big|\Gamma_1^{-1/\alpha} \frac{|f_t(U_1^{(n)})|}{\max_{0 \leq u \leq (n-1)\mathbf{1}} |f_u(U_1^{(n)})|}\Big| > c_n x\Big) \\
&\quad = \frac{1}{2} P\Big(\Gamma_1^{-1/\alpha} > \frac{c_n}{b_n} x\Big) \to \frac{1}{2}
\end{aligned}
$$

as $n \to \infty$. Hence lack of tightness follows.

Suppose now that (2.6.7) holds. Let K be a positive integer such that

$$\alpha(K+1)\theta > d. \tag{2.6.14}$$

We claim that, in this case, for all $\epsilon > 0$ satisfying

$$0 < \epsilon < \frac{1}{K}, \tag{2.6.15}$$

we have

$$P\Big(\max_{0 \leq t \leq (n-1)\mathbf{1}} |X_t| > \lambda b_n,\ \Gamma_1^{-1/\alpha} \leq \epsilon\lambda\Big) \to 0 \quad \text{ as } n \to \infty \tag{2.6.16}$$

for all $\lambda > 0$. Indeed, choose

$$\frac{d}{\theta} < p < \alpha(K+1). \tag{2.6.17}$$

Notice that the probability in the left-hand side of (2.6.16) is bounded from above by

$$\sum_{0 \leq t \leq (n-1)\mathbf{1}} P\Big(|X_t| > \lambda b_n,\ \Gamma_j^{-1/\alpha} \frac{|f_t(U_j^{(n)})|}{\max\limits_{0 \leq u \leq (n-1)\mathbf{1}} |f_u(U_j^{(n)})|} \leq \epsilon\lambda \ \text{ for all } j = 1, 2, \ldots\Big).$$

For f in (2.3.3), let

$$\|f\|_\alpha = \Big(\int_S |f(s)|^\alpha \mu(ds)\Big)^{1/\alpha},$$

and notice that, for any $t \in [0, (n-1)\mathbf{1}]$, the points

$$b_n \varepsilon_j \Gamma_j^{-1/\alpha} \frac{|f_t(U_j^{(n)})|}{\max_{0 \leq u \leq (n-1)\mathbf{1}} |f_u(U_j^{(n)})|}, \qquad j = 1, 2, \ldots$$

represent a symmetric Poisson random measure on $\mathbb{R}$ whose mean measure assigns a mass of $x^{-\alpha}\|f\|_\alpha^\alpha/2$ to the set (x,∞) for every $x>0$ (see Propositions 4.3.1 and 4.4.1 of Resnick (1992))). Since the same random measure can be represented by the points

$$\varepsilon_j\Gamma_j^{-1/\alpha}\|f\|_\alpha, \qquad j=1,2,\ldots,$$

we conclude that the probability in (2.6.16) is bounded from above by

$$\begin{aligned}
&n^d P\Big(C_\alpha^{1/\alpha}\Big|\sum_{j=1}^{\infty}\varepsilon_j\Gamma_j^{-1/\alpha}\Big| > \lambda\|f\|_\alpha^{-1}b_n,\ \Gamma_j^{-1/\alpha}\le \epsilon\lambda\|f\|_\alpha^{-1}b_n \text{ for all } j\ge 1\Big)\\
&\le n^d P\Big(C_\alpha^{1/\alpha}\Big|\sum_{j=K+1}^{\infty}\varepsilon_j\Gamma_j^{-1/\alpha}\Big| > \lambda(1-\epsilon K)\|f\|_\alpha^{-1}b_n\Big)\\
&\le n^d b_n^{-p}\frac{\|f\|_\alpha^p E|C_\alpha^{1/\alpha}\sum_{j=K+1}^{\infty}\varepsilon_j\Gamma_j^{-1/\alpha}|^p}{\lambda^p(1-\epsilon K)^p}\,.
\end{aligned}$$

As long as the expectation above is finite, the latter expression goes to 0 as $n\to\infty$ and hence (2.6.16) follows. The expectation is finite by the choice of p in (2.6.17). Indeed, notice that $E\,\Gamma_j^{-p/\alpha}<\infty$ for all $j\ge K+1$ and that, by the Stirling formula, $E\,\Gamma_j^{-p/\alpha}\sim e^{p/\alpha}j^{-p/\alpha}$ as $j\to\infty$. Assuming, without loss of generality, that $p/2=m$ is an integer (since we can always increase K and get such a p), we see that for finite positive constants c_1, c_2,

$$\begin{aligned}
E\Big|\sum_{j=K+1}^{\infty}\varepsilon_j\Gamma_j^{-1/\alpha}\Big|^p &\le c_1 E\Big(\sum_{j=K+1}^{\infty}\Gamma_j^{-2/\alpha}\Big)^{p/2}\\
&= c_1\sum_{j_1=K+1}^{\infty}\cdots\sum_{j_m=K+1}^{\infty}E\prod_{i=1}^{m}\Gamma_{j_i}^{-2/\alpha}\\
&\le c_1\Big(\sum_{j=K+1}^{\infty}(E\,\Gamma_j^{-2m/\alpha})^{1/m}\Big)^m\\
&= c_1\Big(\sum_{j=K+1}^{\infty}(E\,\Gamma_j^{-p/\alpha})^{2/p}\Big)^{p/2}\\
&\le c_2\Big(\sum_{j=K+1}^{\infty}j^{-2/\alpha}\Big)^{p/2}<\infty\,.
\end{aligned}$$

Fix $\epsilon>0$ satisfying (2.6.15). Given $\delta>0$, choose $\lambda>0$ such that $P(\Gamma_1^{-1/\alpha}>\epsilon\lambda)\le$

$\delta/2$, choose n_0 such that

$$P\Big(\max_{0\le t\le (n-1)\mathbf{1}} |X_t| > \lambda b_n,\, \Gamma_1^{-1/\alpha} \le \epsilon\lambda\Big) \le \frac{\delta}{2} \qquad \text{for } n > n_0$$

and $\lambda' \le \lambda$ such that

$$P(M_k > \lambda' b_k) \le \delta \qquad \text{for } k = 1, \ldots, n_0\,.$$

Then

$$P\Big(\frac{1}{b_n} M_n > \lambda'\Big) \le \delta$$

for all $n \ge 1$, and so (2.6.6) holds with $\zeta_n \equiv 1$.

Now, suppose that $\mathbf{X}$ is generated by a conservative action. Let $\mathbf{Y}$ be a stationary $S\alpha S$ random field independent of $\mathbf{X}$, also given by an integral representation of the form (2.3.1), say,

$$Y_t = \int_{S'} g_t(s) M'(ds), \qquad t \in \mathbb{Z}^d,$$

where M' is a $S\alpha S$ random measure with control measure μ', independent of M in the integral representation of $\mathbf{X}$, with the functions g_t also given in the form (2.3.3), with some nonsingular measure preserving conservative action ϕ' on S' and such that

$$b_n^Y = \Big(\int_{S'} \max_{0\le t\le (n-1)\mathbf{1}} |g_t(s)|^\alpha \mu'(ds)\Big)^{1/\alpha}, \qquad t \in \mathbb{Z}^d,$$

satisfies (2.6.7) for some $\theta > 0$. Random fields $\mathbf{Y}$ with above properties exist (see the Example 2.8.2). However, the above step may require enlarging the probability space we are working with. Let $\mathbf{Z} = \mathbf{X} + \mathbf{Y}$. Then $\mathbf{Z}$ is a stationary $S\alpha S$ random field generated by a conservative $\mathbb{Z}^d$-action. We use its natural integral representation on $S \cup S'$ with the naturally defined action on that space. Let b_n^Z be the corresponding quantity in (2.5.2) defined for the process $\mathbf{Z}$. Note that $b_n^Z \ge b_n^Y$ for all n, hence the random field $\mathbf{Z}$ satisfies (2.6.7) as well. By the already proven part of the theorem, the sequence $(b_n^Z)^{-1} \max_{0\le t\le (n-1)\mathbf{1}} |Z_t|$, $n = 1, 2, \ldots$, is tight. Since for any $x > 0$ and $n = 1, 2, \ldots$,

$$P\Big(\max_{0\le t\le (n-1)\mathbf{1}} |Z_t| > x\Big) \ge \frac{1}{2} P\Big(\max_{0\le t\le (n-1)\mathbf{1}} |X_t| > x\Big)$$

by the symmetry of $\mathbf{Y}$, we conclude that the sequence of random variables $(b_n^Z)^{-1} \max_{0\le t\le (n-1)\mathbf{1}} |X_t|$, $n = 1, 2, \ldots$, is tight as well. However, the random field $\mathbf{Z}$ is generated by a conservative action and hence, by Proposition 2.5.1, $b_n^Z = o(n^{d/\alpha})$. Therefore, (2.6.4) follows.

Suppose now that (2.6.9) holds. Then for every $1 \le j_1 < j_2$ and $\epsilon > 0$

$$\begin{aligned} &P\Big(\text{for some } t \in [0, (n-1)\mathbf{1}],\ \Gamma_{j_i}^{-1/\alpha} \frac{|f_t(U_{j_i}^{(n)})|}{\max\limits_{0\le u\le (n-1)\mathbf{1}} |f_u(U_{j_i}^{(n)})|} > \epsilon,\ i = 1, 2\Big) \\ &\le P(\Gamma_1 \le \tau) + P\Big(\text{for some } t \in [0, (n-1)\mathbf{1}], \\ &\qquad \frac{|f_t(U_j^{(n)})|}{\max_{0\le u\le (n-1)\mathbf{1}} |f_u(U_j^{(n)})|} > \epsilon\tau^{1/\alpha},\ j = 1, 2\Big) \end{aligned}$$

for any $\tau > 0$. Letting first $n \to \infty$ and then $\tau \to 0$ shows that, for every $1 \le j_1 < j_2$ and $\epsilon > 0$,

$$\begin{aligned} &\lim_{n\to\infty} P\Big(\text{for some } t \in [0, (n-1)\mathbf{1}], \\ &\qquad \Gamma_{j_i}^{-1/\alpha} \frac{|f_t(U_{j_i}^{(n)})|}{\max_{0\le u\le (n-1)\mathbf{1}} |f_u(U_{j_i}^{(n)})|} > \epsilon,\ i = 1, 2\Big) = 0 \end{aligned} \tag{2.6.18}$$

Observe, further, that for any $\epsilon > 0$,

$$\begin{aligned} &P\Big(\text{for some } t \in [0, (n-1)\mathbf{1}], \\ &\qquad \Gamma_j^{-1/\alpha} \frac{|f_t(U_j^{(n)})|}{\max_{0\le u\le (n-1)\mathbf{1}} |f_u(U_j^{(n)})|} > \epsilon,\ \text{for at least 2 different } j\Big) \\ &=: \phi_n^{(1)}(\epsilon) \le P(\Gamma_J^{-1/\alpha} > \epsilon) \\ &\quad + \sum_{j_1=1}^{J-1} \sum_{j_2=j_1+1}^{J-1} P\Big(\text{for some } t \in [0, (n-1)\mathbf{1}], \\ &\qquad \Gamma_{j_i}^{-1/\alpha} \frac{|f_t(U_{j_i}^{(n)})|}{\max_{0\le u\le (n-1)\mathbf{1}} |f_u(U_{j_i}^{(n)})|} > \epsilon,\ i = 1, 2\Big) \end{aligned}$$

for any $J = 1, 2, \ldots$. Letting $n \to \infty$ and using (2.6.18), and then letting $J \to \infty$ shows that, for every $\epsilon > 0$

$$\lim_{n\to\infty} \phi_n^{(1)}(\epsilon) = 0. \tag{2.6.19}$$

Suppose now that both (2.6.7) and (2.6.9) hold. Let K be as in (2.6.14). Let $\epsilon > 0$ and $0 < \delta < 1$ satisfy

$$0 < \epsilon < \frac{\delta}{K}. \tag{2.6.20}$$

For any $\lambda > 0$, we have

$$\begin{aligned}
&P\Big(\frac{1}{b_n} M_n > \lambda\Big) \\
&\quad \le P(C_\alpha^{1/\alpha}\Gamma_1^{-1/\alpha} > \lambda(1-\delta)) + \phi_n^{(1)}(C_\alpha^{-1/\alpha}\epsilon\lambda) \\
&\qquad + P\Big(\max_{0\le t\le (n-1)\mathbf{1}} \Big|\sum_{j=1}^{\infty} \varepsilon_j \Gamma_j^{-1/\alpha} \frac{|f_t(U_j^{(n)})|}{\max\limits_{0\le u\le (n-1)\mathbf{1}} |f_u(U_j^{(n)})|}\Big| > C_\alpha^{-1/\alpha}\lambda, \\
&\qquad\qquad \Gamma_1^{-1/\alpha} \le C_\alpha^{-1/\alpha}\lambda(1-\delta), \text{ and for each t } \in [0, (n-1)\mathbf{1}], \\
&\qquad\qquad \Gamma_j^{-1/\alpha} \frac{|f_t(U_j^{(n)})|}{\max_{0\le u\le (n-1)\mathbf{1}} |f_u(U_j^{(n)})|} > C_\alpha^{-1/\alpha}\epsilon\lambda \\
&\qquad\qquad\qquad \text{for at most one } j = 1, 2, \ldots\Big) \\
&\quad =: P(C_\alpha^{1/\alpha}\Gamma_1^{-1/\alpha} > \lambda(1-\delta)) + \phi_n^{(1)}(C_\alpha^{-1/\alpha}\epsilon\lambda) + \phi_n^{(2)}(\epsilon, \delta).
\end{aligned} \tag{2.6.21}$$

Proceeding similarly to the argument used in proving (2.6.16), we have

$$\begin{aligned}
&\phi_n^{(2)}(\epsilon, \delta) \\
&\quad \le \sum_{0\le t\le (n-1)\mathbf{1}} P\Big(\Big|\sum_{j=1}^{\infty} \varepsilon_j \Gamma_j^{-1/\alpha} \frac{|f_t(U_j^{(n)})|}{\max\limits_{0\le u\le (n-1)\mathbf{1}} |f_u(U_j^{(n)})|}\Big| > C_\alpha^{-1/\alpha}\lambda, \\
&\qquad\qquad \Gamma_j^{-1/\alpha} \frac{|f_t(U_j^{(n)})|}{\max_{0\le u\le (n-1)\mathbf{1}} |f_u(U_j^{(n)})|} \\
&\qquad\qquad \le C_\alpha^{-1/\alpha}\lambda(1-\delta) \text{ for each } j = 1, 2, \ldots \\
&\qquad\qquad \text{and } \Gamma_j^{-1/\alpha} \frac{|f_t(U_j^{(n)})|}{\max\limits_{0\le u\le (n-1)\mathbf{1}} |f_u(U_j^{(n)})|} > C_\alpha^{-1/\alpha}\epsilon\lambda \\
&\qquad\qquad\qquad \text{for at most one } j = 1, 2, \ldots\Big)
\end{aligned} \tag{2.6.22}$$

$$
\begin{aligned}
&\le n^d P\Big(\Big| \sum_{j=1}^{\infty} \varepsilon_j \Gamma_j^{-1/\alpha} \Big| > C_\alpha^{-1/\alpha} \lambda \|f\|_\alpha^{-1} b_n, \\
&\qquad\qquad \Gamma_1^{-1/\alpha} \le C_\alpha^{-1/\alpha} \lambda (1-\delta) \|f\|_\alpha^{-1} b_n, \\
&\qquad\qquad\quad \text{and } \Gamma_2^{-1/\alpha} \le C_\alpha^{-1/\alpha} \epsilon \lambda \|f\|_\alpha^{-1} b_n \Big)
\end{aligned}
$$

and the latter expression goes to zero as $n \to \infty$ by the choice of ϵ and δ, as in the proof of (2.6.16). We conclude by (2.6.19) and (2.6.22) that, for any $0 < \delta < 1$,

$$
\begin{aligned}
\limsup_{n\to\infty} P\Big(\frac{1}{b_n} M_n > \lambda \Big) &\le P(C_\alpha^{1/\alpha} \Gamma_1^{-1\alpha} > \lambda(1-\delta)) \\
&= 1 - \exp\{-C_\alpha \lambda^{-\alpha} (1-\delta)^{-\alpha}\},
\end{aligned}
$$

and letting $\delta \to 0$, we obtain

$$
\limsup_{n\to\infty} P\Big(\frac{1}{b_n} M_n > \lambda \Big) \le 1 - \exp\{-C_\alpha \lambda^{-\alpha}\}. \tag{2.6.23}
$$

In the opposite direction, the argument is similar. For any ϵ and $\delta > 0$ satisfying (2.6.20), we have

$$
\begin{aligned}
&P\Big(\frac{1}{b_n} M_n > \lambda \Big) \\
&\ge P(C_\alpha^{1/\alpha} \Gamma_1^{-1/\alpha} > \lambda(1+\delta)) - \phi_n^{(1)}(C_\alpha^{-1/\alpha} \epsilon \lambda) \\
&\quad - P\Big(\max_{0 \le t \le (n-1)\mathbf{1}} \Big| \sum_{j=1}^{\infty} \varepsilon_j \Gamma_j^{-1/\alpha} \frac{|f_t(U_j^{(n)})|}{\max_{0\le u \le (n-1)\mathbf{1}} |f_u(U_j^{(n)})|} \Big| \le C_\alpha^{-1/\alpha} \lambda, \\
&\qquad\quad \Gamma_1^{-1/\alpha} > C_\alpha^{-1/\alpha} \lambda(1+\delta), \text{and for all } t \in [0, (n-1)\mathbf{1}], \\
&\qquad\quad \Gamma_j^{-1/\alpha} \frac{|f_t(U_j^{(n)})|}{\max_{0\le u \le (n-1)\mathbf{1}} |f_u(U_j^{(n)})|} > C_\alpha^{-1/\alpha} \epsilon \lambda \\
&\qquad\qquad\qquad\qquad\qquad \text{for at most one } j = 1, 2, \ldots \Big) \\
&=: P(C_\alpha^{1/\alpha} \Gamma_1^{-1/\alpha} > \lambda(1+\delta)) - \phi_n^{(1)}(C_\alpha^{-1/\alpha} \epsilon \lambda) - \phi_n^{(3)}(\epsilon, \delta).
\end{aligned}
$$

Once again, the choice of ϵ and δ gives us

$$
\lim_{n\to\infty} \phi_n^{(3)}(\epsilon, \delta) = 0, \tag{2.6.24}
$$

and so we conclude by (2.6.19) and (2.6.24), that for any $\delta > 0$,

$$\begin{aligned}\liminf_{n\to\infty} P\Big(\frac{1}{b_n} M_n > \lambda\Big) &\geq P(C_\alpha^{1/\alpha}\Gamma_1^{-1/\alpha} > \lambda(1+\delta)) \\ &= 1 - \exp\{-C_\alpha \lambda^{-\alpha}(1+\delta)^{-\alpha}\},\end{aligned}$$

and letting $\delta \to 0$, we obtain a lower bound matching (2.6.23). This proves (2.6.10).

If $\mathbf{X}$ is not generated by a conservative action (i.e., the component $X^{\mathcal{D}}$ in (2.3.8) generated by a dissipative action is nonzero) then it follows from Proposition 2.5.1 that

$$n^{-d/\alpha} b_n \to K_X \qquad \text{as } n \to \infty.$$

In particular, both conditions (2.6.7) and (2.6.9) are satisfied (see Remark 2.6.2). Therefore, (2.6.1) follows from the already proven (2.6.10), and the proof of all parts of the theorem is complete. □

2.7 Connections with Group Theory

When the underlying action is not conservative Theorem 2.6.1 yields the exact rate of growth of M_n. For conservative actions, however, the actual rate of growth of the partial maxima depends on further properties of the action. In this section we investigate the effect of the group theoretic structure of the action on the rate of growth of the partial maximum. We start with introducing the appropriate notation.

Consider $A := \{\phi_t : t \in \mathbb{Z}^d\}$ as a subgroup of the group of invertible nonsingular transformations on (S, μ) and define a group homomorphism

$$\Phi : \mathbb{Z}^d \to A$$

by $\Phi(t) = \phi_t$ for all $t \in \mathbb{Z}^d$. Let $K := Ker(\Phi) = \{t \in \mathbb{Z}^d : \phi_t = 1_S\}$, where 1_S denote the identity map on S. Then K is a free abelian group and by first isomorphism theorem of groups (see, for example, Lang (2002)) we have

$$A \simeq \mathbb{Z}^d / K \,.$$

Hence, by Theorem 8.5 in Chapter I of Lang (2002), we get

$$A = \bar{F} \oplus \bar{N} \,,$$

where $\bar{F}$ is a free abelian group and $\bar{N}$ is a finite group. Assume $rank(\bar{F}) = p \geq 1$ and $|\bar{N}| = l$. Since $\bar{F}$ is free, there exists an injective group homomorphism

$$\Psi : \bar{F} \to \mathbb{Z}^d$$

such that $\Phi \circ \Psi = 1_{\bar{F}}$. Let $F = \Psi(\bar{F})$. Then F is a free subgroup of $\mathbb{Z}^d$ of rank p. In particular, $p \leq d$.

The rank p is the effective dimension of the random field, giving more precise information on the rate of growth of the partial maximum than the nominal dimension d. We start with showing that this is true for the sequence $\{b_n\}$ in (2.5.2).

Proposition 2.7.1. *Let* $\{f_t\}$ *be given by* (2.3.3)*. Assume that* (2.3.2) *holds. Then we have the following:*

1. *If* $\{\phi_t\}_{t \in F}$ *is conservative then*

$$n^{-p/\alpha} b_n \to 0 \,. \tag{2.7.1}$$

2. *If* $\{\phi_t\}_{t \in F}$ *is dissipative then*

$$n^{-p/\alpha} b_n \to a \tag{2.7.2}$$

   for some $a \in (0, \infty)$.

Proof. 1. It is easy to check that $F \cap K = \{0\}$ and hence the sum $F + K$ is direct. Suppose $G = F \oplus K$. Using group isomorphism theorems we have

$$\mathbb{Z}^d / G \simeq (\mathbb{Z}^d / K)/(F \oplus K/K) \simeq A/\bar{F} \simeq \bar{N} \,.$$

Assume that $x_1 + G,\ x_2 + G,\ \ldots, x_l + G$ are all the cosets of G in $\mathbb{Z}^d$. Let $rank(K) = q$. Choose a basis $\{u_1, u_2, \ldots, u_p\}$ of F and a basis $\{v_1, v_2, \ldots, v_q\}$ of K. We need the following

Lemma 2.7.2. *There are positive integers c and N such that for every $n \geq 1$*

$$[-n\mathbf{1}, n\mathbf{1}] \subseteq \bigcup_{k=1}^{l} (x_k + G_{cn}) \tag{2.7.3}$$

where for $m \geq 1$

$$G_m := \{\sum_{i=1}^{p} \alpha_i u_i + \sum_{j=1}^{q} \beta_j v_j :\ |\alpha_i|, |\beta_j| \leq m \text{ for all } i, j\}\,.$$

Proof. Let $r = p + q$. For ease of notation we define

$$w_i = \begin{cases} u_i & 1 \leq i \leq p\,, \\ v_{i-p} & p+1 \leq i \leq r\,. \end{cases}$$

Then $\{w_1, w_2, \ldots, w_r\}$ is a basis for G. We establish (2.7.3) in two steps as follows:

Step 1. There is an integer $c' \geq 1$ such that

$$[-n\mathbf{1}, n\mathbf{1}] \cap G \subseteq G_{c'n} \quad \text{for all } n \geq 1\,.$$

Proof of Step 1. Take $y \in [-n\mathbf{1}, n\mathbf{1}] \cap G$. Then, $y = \eta_1 w_1 + \eta_2 w_2 + \cdots + \eta_r w_r$ for some $\eta_1, \eta_2, \ldots, \eta_r \in \mathbb{Z}$. We have to show $|\eta_i| \leq c'n$ for all $1 \leq i \leq r$ for some $c' \geq 1$ that does not depend on n. Let $\widetilde{\eta}^T := (\eta_1, \eta_2, \ldots, \eta_r) \in \mathbb{Z}^r$. Then,

$$y = W\widetilde{\eta} \tag{2.7.4}$$

where W is the $d \times r$ matrix with w_i as the i^{th} column. The columns of W are linearly independent over $\mathbb{Z}$ and hence over $\mathbb{R}$. Hence there is a $r \times d$ matrix Z such that

$$ZW = I$$

where I is the identity matrix of order r. Hence from (2.7.4) we have

$$\widetilde{\eta} = Zy\,.$$

For all $1 \leq i \leq r$ we get

$$|\eta_i| \;\leq\; \|\widetilde{\eta}\| \;\leq\; \|Z\|\|y\| \;\leq\; \|Z\|n\sqrt{d} \;\leq\; c'n$$

where $c' = [\|Z\|\sqrt{d}] + 1$. This proves Step 1.

Step 2. Let

$$M = \max_{1\leq k\leq l} \|x_k\|_\infty + 1$$

where $\|\cdot\|_\infty$ denotes the sup-norm on R^d, and $c = c'M$. Then for all $n \geq 1$ we have

$$[-n\mathbf{1}, n\mathbf{1}] \subseteq \bigcup_{k=1}^{l}(x_k + G_{cn})\,.$$

Proof of Step 2. Take $y \in [-n\mathbf{1}, n\mathbf{1}]$. Then $y \in x_{k_0} + G$ for some $1 \leq k_0 \leq l$. Clearly,

$$y' := y - x_{k_o} \in [-(n + M - 1)\mathbf{1}, (n + M - 1)\mathbf{1}] \cap G\,.$$

By Step 1, $y' \in G_{c'(n+M-1)} \subseteq G_{cn}$, and hence, $y \in x_{k_0} + G_{cn} \subseteq \bigcup_{k=1}^{l}(x_k + G_{cn})$, proving Step 2 and the lemma. □

To prove the first part of Proposition 2.7.1 from Lemma 2.7.2 define, for $k = 1, 2, \ldots, l$,

$$g_k = f \circ \phi_{x_k}\Big(\frac{d\mu \circ \phi_{x_k}}{d\mu}\Big)^{1/\alpha}\,.$$

Then for $t = x_k + \sum_{i=1}^{p} \alpha_i u_i + \sum_{j=1}^{q} \beta_j v_j$ we have

$$|f_t(s)| = |g_k \circ \phi_{\sum_{i=1}^{p} \alpha_i u_i}(s)|\Big(\frac{d\mu \circ \phi_{\sum_{i=1}^{p} \alpha_i u_i}}{d\mu}(s)\Big)^{1/\alpha}\,. \tag{2.7.5}$$

Using Lemma 2.7.2, (2.7.5), and the stationarity of the field, it follows, for all $n \geq 1$, that

$$
\begin{aligned}
b_n^\alpha &\leq b_{2n+1}^\alpha = \int_S \max_{-n\mathbf{1} \leq t \leq n\mathbf{1}} |f_t(s)|^\alpha \mu(ds) \\
&\leq \int_S \max_{1\leq k\leq l} \max_{|\alpha_i|\leq cn} \left(|g_k \circ \phi_{\sum_{i=1}^p \alpha_i u_i}(s)|^\alpha \frac{d\mu \circ \phi_{\sum_{i=1}^p \alpha_i u_i}}{d\mu}(s) \right) \mu(ds) \\
&\leq \sum_{k=1}^{l} \int_S \max_{|\alpha_i|\leq cn} \left(|g_k \circ \phi_{\sum_{i=1}^p \alpha_i u_i}(s)|^\alpha \frac{d\mu \circ \phi_{\sum_{i=1}^p \alpha_i u_i}}{d\mu}(s) \right) \mu(ds) \\
&= o(n^p)\,.
\end{aligned}
$$

The last step follows from Proposition 2.5.1 and Remark 2.5.2.

2. We start this proof with the following combinatorial fact:

Lemma 2.7.3. *For $n \geq 1$ and $k = 1, 2, \ldots, l$, let*

$$F_{k,n} = \{u \in x_k + F : \text{ there exists } v \in K \text{ such that } u + v \in [-n\mathbf{1}, n\mathbf{1}]\}\,.$$

Then there is a positive real number $\mathcal{V}$ such that for all $k = 1, 2, \ldots, l$,

$$\lim_{n\to\infty} \frac{|F_{k,n}|}{n^p} = \mathcal{V}\,. \tag{2.7.6}$$

Here $|A|$ stands for the cardinality of a set A.

Proof. One of $F_{k,n}$ is the set

$$F_n = \{y \in F : y + v \in [-n\mathbf{1}, n\mathbf{1}] \text{ for some } v \in K\}\,. \tag{2.7.7}$$

Firstly, we will show

$$\lim_{n\to\infty} \frac{|F_n|}{n^p} = \mathcal{V} \tag{2.7.8}$$

for some $\mathcal{V} > 0$. To show this let W be the matrix used in the proof of Lemma 2.7.2. We can partition W into two submatrices as follows:

$$W = [\, U \,|\, V \,]$$

where U is the $d \times p$ matrix whose i^{th} column is u_i and V is the $d \times q$ matrix whose j^{th} column is v_j. Since the columns of U are linearly independent over $\mathbb{Z}$, we have

$$|F_n| = |\{\alpha \in \mathbb{Z}^p : \text{there exists}\, \beta \in \mathbb{Z}^q \text{ such that } \|U\alpha + V\beta\|_\infty \leq n\}|\,.$$

Let $P := \{x \in \mathbb{R}^r : \|Wx\|_\infty \leq 1\}$ and $\pi : \mathbb{R}^r \to \mathbb{R}^p$ denote the projection map on the first p coordinates:

$$\pi(x_1, x_2, \ldots, x_r) = (x_1, x_2, \ldots, x_p)\,.$$

Then we have

$$\frac{|F_n|}{n^p} = \frac{|\pi(\mathbb{Z}^r \cap nP)|}{n^p} =: a_n\,.$$

Let

$$b_n := \frac{|\mathbb{Z}^p \cap n\pi(P)|}{n^p}\,.$$

Clearly, $a_n \leq b_n$. Since P is a rational polytope (i.e., a polytope whose vertices have rational coordinates) so is $\pi(P)$. Hence, by Theorem 1 of De Loera (2005), it follows that

$$\limsup_{n\to\infty} a_n \leq \lim_{n\to\infty} b_n = \mathcal{V} \tag{2.7.9}$$

where $\mathcal{V} = Volume(\pi(P))$, the p-dimensional volume of $\pi(P)$. This volume is positive since the latter set, obviously, contains a small ball centered at the origin. For the other inequality we let

$$P_m := \left\{x \in \mathbb{R}^r : \|Wx\|_\infty \leq 1 - \frac{\|W\|_\infty}{m}\right\}$$

where $\|W\|_\infty := \sup_{x\neq 0} \frac{\|Wx\|_\infty}{\|x\|_\infty} \in \mathbb{Z}$ since W is a matrix with integer entries. Hence for all $m > \|W\|_\infty$, P_m is a rational polytope of dimension r. Also, $P_m \uparrow P$. Now fix $m > \|W\|_\infty$. Observe that

$$\left\{y \in \mathbb{R}^r : \|y - x\|_\infty \leq \frac{1}{m}\right\} \subseteq P \quad \text{for all } x \in P_m\,.$$

Hence, it follows that for all $n > m$,

$$\pi\left(\frac{1}{n}\mathbb{Z}^r \cap P\right) \supseteq \frac{1}{n}\mathbb{Z}^p \cap \pi(P_m)\,,$$

which, along with Theorem 1 of De Loera (2005), implies

$$\liminf_{n\to\infty} a_n \geq \lim_{n\to\infty} \frac{|\mathbb{Z}^p \cap n\pi(P_m)|}{n^p} = \mathcal{V}_m \tag{2.7.10}$$

where $\mathcal{V}_m = Volume(\pi(P_m))$ is, once again, the p-dimensional volume. Since $P_m \uparrow P$, it follows that $\mathcal{V}_m \uparrow \mathcal{V}$. Hence (2.7.8) follows from (2.7.9) and (2.7.10).

Now fix $k = 1, 2, \ldots, l$ and let $M = \|x_k\|$. Observe that for all $n > M$,

$$|F_{n-M}| \leq |F_{k,n}| \leq |F_{n+M}|\,.$$

Hence (2.7.6) follows from (2.7.8). □

We now return to the proof of the second part of the proposition. We give a group structure to

$$H := \bigcup_{k=1}^{l} (x_k + F) \tag{2.7.11}$$

as follows. For all $u_1, u_2 \in H$, there exists unique $u \in H$ such that $(u_1 + u_2) - u \in K$. We define this u to be $u_1 \oplus u_2$. It is not hard to check that $(H, \oplus)$ is a countable abelian group. In fact, $H \simeq \mathbb{Z}^d/K$. We can define a nonsingular group action $\{\psi_u\}$ of H on S as

$$\psi_u = \phi_u \quad \text{for all } u \in H\,. \tag{2.7.12}$$

Notice that if $h \in L^1(S,\mu), h > 0$, then, since (2.7.11) is a disjoint union,

$$\sum_{u\in H} \frac{d\mu\circ\psi_u}{d\mu} h\circ\psi_u = \sum_{t\in F} \frac{d\mu\circ\phi_t}{d\mu} \tilde{h}\circ\phi_t\,, \tag{2.7.13}$$

where

$$\tilde{h} = \sum_{k=1}^{l} \frac{d\mu\circ\phi_{x_k}}{d\mu} h\circ\phi_{x_k}\,.$$

Clearly $\tilde{h} \in L^1(S,\mu)$ and $\tilde{h} > 0$. Hence using Corollary 2.2.4 and the dissipativity of $\{\phi_t\}_{t\in F}$, we see that the second sum in (2.7.13) is finite almost everywhere. Another appeal to Corollary 2.2.4 shows that $\{\psi_u\}_{u\in H}$ is a dissipative group action.

Define a random field $\{Y_u\}_{u\in H}$ as

$$Y_u = \int_S \tilde{f}_u(s) M(ds), \quad u \in H, \tag{2.7.14}$$

where

$$\tilde{f}_u = f \circ \psi_u \left(\frac{d\mu \circ \psi_u}{d\mu}\right)^{1/\alpha} \quad u \in H.$$

Clearly $\{Y_u\}_{u\in H}$ is an H-stationary $S\alpha S$ random field generated by the dissipative action $\{\psi_u\}_{u\in H}$. Hence, by Remark 2.4.2, there is a standard Borel space $(W, \mathcal{W})$ with a σ-finite measure ν on it such that

$$Y_u \stackrel{d}{=} \int_{W\times H} g(w, u \oplus s)\, N(dw, ds) \quad u \in H,$$

for some $g \in L^\alpha(W \times H, \nu \otimes \tau)$, where τ is the counting measure on H, and N is a $S\alpha S$ random measure on $W \times H$ with control measure $\nu \otimes \tau$.

Let, for all $w \in W$,

$$g^*(w) := \sup_{u\in H} |g(w, u)|\,. \tag{2.7.15}$$

Then, clearly, $g^* \in L^\alpha(W, \nu)$. We will show that (2.7.2) holds with

$$a := \left(\frac{\mathcal{V}l}{2^p} \int_W (g^*(w))^\alpha d\nu(w)\right)^{1/\alpha} \in (0, \infty)\,. \tag{2.7.16}$$

Since b_n is an increasing sequence, it is enough to show

$$\lim_{n\to\infty} \frac{b_{2n+1}}{(2n+1)^{p/\alpha}} = a\,. \tag{2.7.17}$$

Let $H_n := \bigcup_{k=1}^{l} F_{k,n}$. Then by stationarity of $\{X_t\}_{t\in\mathbb{Z}^d}$ we have for all $n \geq 1$,

$$\begin{aligned} b_{2n+1}^\alpha &= \int_S \max_{-n\mathbf{1}\leq t\leq n\mathbf{1}} |f_t(s)|^\alpha \mu(ds) \\ &= \int_S \max_{u\in H_n} |\tilde{f}_u(s)|^\alpha \mu(ds) \\ &= \sum_{s\in H} \int_W \max_{u\in H_n} |g(w, s\oplus u)|^\alpha \nu(dw)\,. \end{aligned} \tag{2.7.18}$$

The last equality follows from Corollary 4.4.6 of Samorodnitsky and Taqqu (1994). We define a map $N : H \to \{0, 1, \ldots\}$ as,

$$N(u) := \min\{\|u + v\|_\infty : v \in K\}\,.$$

Clearly, for all $u \in H$,

$$N(u^{-1}) = N(u)\,, \tag{2.7.19}$$

where u^{-1} is the inverse of u in H. Also, $N(\cdot)$ satisfies the following "triangle inequality": for all $u_1, u_2 \in H$,

$$N(u_1 \oplus u_2) \le N(u_1) + N(u_2)\,. \tag{2.7.20}$$

Observe that

$$H_n = \{u \in H : N(u) \le n\}\,. \tag{2.7.21}$$

From Lemma 2.7.2 we have H_n's are finite and Lemma 2.7.3 yields

$$|H_n| \sim \mathcal{V} l n^p\,. \tag{2.7.22}$$

Also, clearly, $H_n \uparrow H$. As in the proof of (2.5.7), we first assume g has compact support, i.e., $g(w,u) I_{W \times H_m^c}(w,u) = 0$ for some $m \ge 1$. Then using (2.7.19) and (2.7.20), the expression in (2.7.18) becomes

$$\begin{aligned} b_{2n+1}^\alpha &= \sum_{s \in H_{n+m}} \int_W \max_{u \in H_n} |g(w, s \oplus u)|^\alpha \nu(dw) \\ &= \sum_{s \in H_{n-m}} \int_W \max_{u \in H_n} |g(w, s \oplus u)|^\alpha \nu(dw) \\ &\qquad + \sum_{s \in H_{n+m} \cap H_{n-m}^c} \int_W \max_{u \in H_n} |g(w, s \oplus u)|^\alpha \nu(dw) =: A_n + B_n \end{aligned}$$

for all $n > m$. Using (2.7.19) and (2.7.20) once again, we have for all $s \in H_{n-m}$,

$$\max_{u \in H_n} |g(w, s \oplus u)| = g^*(w)\,.$$

Hence, using (2.7.22), we get

$$A_n = |H_{n-m}| \int_W (g^*(w))^\alpha \nu(dw) \sim a^\alpha (2n+1)^p\,,$$

while

$$B_n \le (|H_{n+m}| - |H_{n-m}|) \int_W (g^*(w))^\alpha \nu(dw) = o(n^p) \,.$$

Hence, (2.7.17) follows for g having compact support. The proof in the general case follows by approximating a general kernel g by a kernel with a compact support as done in the proof of (2.5.7). This completes the proof of the proposition. □

The following result sharpens the the description of the asymptotic behavior of the partial maxima of a random field given in Theorem 2.6.1. It reduces to the latter result if $K = \{0\}$.

Theorem 2.7.4. *Let* $\mathbf{X} = \{X_t\}_{t\in\mathbb{Z}^d}$ *be a stationary* $S\alpha S$ *random field, with* $0 < \alpha < 2$*, integral representation* (2.3.1)*, and functions* $\{f_t\}$ *given by* (2.3.3)*. Then, in the terminology introduced in this section, we have the following:*

1. If $\{\phi_t\}_{t\in F}$ *is not conservative then*

$$\frac{1}{n^{p/\alpha}} M_n \Rightarrow c\, Z_\alpha \tag{2.7.23}$$

for some $c \in (0,\infty)$*, and* Z_α *as in* (2.6.3)*. In fact,*

$$c = \left(\frac{\mathcal{V} l C_\alpha}{2^p} \int_W (g^*(w))^\alpha d\nu(w) \right)^{1/\alpha} ,$$

where $\mathcal{V}$ *is given by* (2.7.6)*, while* g^* *is given by* (2.7.15) *applied to the dissipative part of the random field* (2.7.14)*, and* C_α *is as in* (2.6.2)*.*

2. If $\{\phi_t\}_{t\in F}$ *is conservative then*

$$\frac{1}{n^{p/\alpha}} M_n \xrightarrow{p} 0 \,. \tag{2.7.24}$$

Proof. 1. Let r_n be the left hand side of (2.6.9). Then we have

$$\begin{aligned} r_n &\leq P\left(\text{for some } u \in H_n, \frac{|f_u(U_j^{(n)})|}{\max_{s \in H_n} |f_s(U_j^{(n)})|} > \epsilon,\ j = 1, 2\right) \\ &\leq |H_n| \left(\epsilon^{-\alpha} b_n^{-\alpha} \int_S |f(s)|^\alpha \mu(ds)\right)^2 . \end{aligned} \tag{2.7.25}$$

The inequality (2.7.25) follows using the argument given in Remark (2.6.2). Since $\{\phi_t\}_{t \in F}$ is not conservative, Proposition 2.7.1 yields that b_n satisfies (2.7.2). Hence, by (2.7.22), we get that (2.6.9) holds in this case. Since b_n satisfies (2.7.2) with a given by (2.7.16), we get (2.7.23) by Theorem 2.6.1.

2. As in the proof of (2.6.4) we can get a stationary $S\alpha S$ random field $\mathbf{Y}$ generated by a conservative $\mathbb{Z}^d$-action such that b_n^Y satisfies (2.6.7) as well as (2.7.1) (this is possible, for instance, by Example 2.8.2 below). Therefore, (2.7.24) follows using the exact same argument as in the proof of (2.6.4). □

Remark 2.7.5. The previous discussion assumes that $p \geq 1$. When $p = 0$ (i.e., when $\mathbb{Z}^d/K$ is a finite group) the random field takes only finitely many different values. Therefore, the sequence M_n remains constant after some stage and so converges to the maximum of finitely many X_t's, not an extreme value random variable.

Remark 2.7.6. Suppose now that $(H, \oplus)$ is any countable abelian group such that there exist a sequence of subsets $\{H_n\}_{n \geq 1}$ of H with $H_n \uparrow H$ satisfying

1. if $u \in H_n$ then $u^{-1} \in H_n$,
2. if $u \in H_m$ and $v \in H_n$ then $u \oplus v \in H_{m+n}$.

Equivalently, H admits a map $N : H \to \{0, 1, \ldots\}$ satisfying (2.7.19) and (2.7.20) so that H_n's are obtained as corresponding "n-balls" defined by (2.7.21). It is not difficult to observe that the phase transition observed in Theorem 2.7.4 holds for stationary $S\alpha S$ random fields indexed by H provided H_n's grow polynomially fast. These are called groups of polynomial volume growth. See, for instance, Gromov (1981) for a discussion of such groups.

2.8 Examples

In this section we consider several examples of stationary $S\alpha S$ random fields associated with conservative flows. As in the one-dimensional case considered in Samorodnitsky (2004a), the idea is to exhibit a variety of possible behaviors in this case.

Our first example shows that, the partial maxima sequence $\{M_n\}$ may not have an extreme value limit. See also Remark 2.7.5 above.

Example 2.8.1. Let $\{Z_t\}_{t\in\mathbb{Z}^d}$ be a collection of i.i.d. standard normal random variables, independent of a positive $(\alpha/2)$-stable random variable A with Laplace transform $E\,e^{-\theta A} = e^{-\theta^{\alpha/2}}$, $\theta \geq 0$. Then $X_t = A^{1/2}Z_t$, $t \in \mathbb{Z}^d$ is a stationary $S\alpha S$ random field, the simplest type of sub-Gaussian $S\alpha S$ random fields (see Section 3.7 in Samorodnitsky and Taqqu (1994)). This random field has an integral representation of the form

$$X_t \stackrel{d}{=} (d_\alpha)^{-1} \int_{\mathbb{R}^{\mathbb{Z}^d}} g_t\, dM, \quad t \in \mathbb{Z}^d \tag{2.8.1}$$

where $d_\alpha = \sqrt{2}(E\,|Z_0|^\alpha)^{1/\alpha}$, and M is a $S\alpha S$ random measure on $\mathbb{R}^{\mathbb{Z}^d}$ whose control measure μ is a probability measure under which the projections g_t, $t \in \mathbb{Z}^d$ are i.i.d. standard normal random variables. Define a $\mathbb{Z}^d$-action $\{\phi_t\}$ as follows,

$$\phi_t((x_s)_{s\in\mathbb{Z}^d}) = (x_{s+t})_{s\in\mathbb{Z}^d} \quad \text{for all } (x_s)_{s\in\mathbb{Z}^d} \in \mathbb{R}^{\mathbb{Z}^d}.$$

Then the integral representation (2.8.1) is of the form (2.3.3), with $c_t \equiv 1$ and the measure preserving action $\{\phi_t\}$. Since $\{\phi_t\}$ is measure preserving action on a finite measure space, it is conservative.

An elementary direct computation shows that

$$b_n^\alpha = (d_\alpha)^{-\alpha}\, E \max_{0\le t\le (n-1)\mathbf{1}} |Z_t|^\alpha \sim (d_\alpha)^{-\alpha}(2d\log n)^{\alpha/2}$$

and

$$\frac{1}{(2d\log n)^{1/2}} \max_{0\le t\le (n-1)\mathbf{1}} |Z_t| \xrightarrow{a.s.} 1$$

as $n \to \infty$. Therefore the assumption (2.6.7) in Theorem 2.6.1 fails, and we see directly that

$$\frac{1}{b_n} M_n \Rightarrow d_\alpha\, A^{1/2}\,,$$

a non-extreme value limit, even though the partial maximum M_n does grow at the same rate as b_n .

The second example exhibits stationary $S\alpha S$ random fields generated by conservative $\mathbb{Z}^d$-actions and satisfying (2.6.7). Note that existence of such a field was needed in the proofs of (2.6.4) and (2.7.24).

Example 2.8.2. Once again, let $\mathbf{X}$ be given by (2.8.1) with $d_\alpha = 1$, but now the control measure μ of the $S\alpha S$ random measure M is a probability measure under which the projections $g_t,\ t \in \mathbb{Z}^d$ are i.i.d. positive Pareto random variables with

$$\mu(g_0 > x) = x^{-\theta} \quad \text{for } x \ge 1$$

for some $\theta > \alpha$. For the same reason as before, $\mathbf{X}$ is generated by a conservative $\mathbb{Z}^d$-action. Note that, for every $p \in (0, \theta)$,

$$\int_{\mathbb{R}^{\mathbb{Z}^d}} \max_{0\le t\le (n-1)\mathbf{1}} |g_t|^p d\mu \ \sim\ c_{p,\theta}\, n^{pd/\theta} \qquad \text{as } n \to \infty \tag{2.8.2}$$

for some finite positive constant $c_{p,\theta}$. Using (2.8.2) with $p = \alpha$ shows that

$$b_n \sim c_{\alpha,\theta}^{1/\alpha}\, n^{d/\theta} \quad \text{as } n \to \infty\,,$$

and so (2.6.7) holds. Furthermore, (2.8.2) with some $p \in (\alpha, \theta)$ shows uniform integrability of the sequence $\{b_n^{-\alpha} \max_{0 \le t \le (n-1)\mathbf{1}} |g_t|^\alpha\}$ with respect to μ. Since (2.6.12) is obvious in this case, we conclude that (2.6.9) holds as well. That is, $n^{-d/\theta} M_n$ converges to an extreme value distribution and hence this example also shows that the rate of growth of M_n can be n^γ for any $\gamma \in (0, d/\alpha)$.

The next example is motivated by a stationary $S\alpha S$ process considered in Mikosch and Samorodnitsky (2000) (see also Example 5.3 in Samorodnitsky (2004a)).

Example 2.8.3. We start with d irreducible aperiodic null-recurrent Markov chains on $\mathbb{Z}$ with laws $P_i^{(1)}(\cdot), P_i^{(2)}(\cdot), \ldots, P_i^{(d)}(\cdot)$, $i \in \mathbb{Z}$ and transition probabilities $(p_{jk}^{(1)}), (p_{jk}^{(2)}), \ldots, (p_{jk}^{(d)})$ respectively. For all $l = 1, 2, \ldots, d$, let $\pi^{(l)} = (\pi_i^{(l)})_{i \in \mathbb{Z}}$ be the σ-finite invariant measure corresponding to the family $(P_i^{(l)})$ satisfying $\pi_0^{(l)} = 1$. Let $\tilde{P}_i^{(l)}$ be the lateral extension of $P_i^{(l)}$ to $\mathbb{Z}^\mathbb{Z}$; that is under $\tilde{P}_i^{(l)}$, $x_0 = i$, $(x_0, x_1, \ldots)$ is a Markov chain with transition probabilities $(p_{jk}^{(l)})$ and $(x_0, x_{-1}, \ldots)$ is a Markov chain with transition probabilities $(\pi_k^{(l)} p_{kj}^{(l)} / \pi_j^{(l)})$. Define a σ-finite measure μ on $S = \mathbb{Z}^\mathbb{Z} \times \mathbb{Z}^\mathbb{Z} \times \cdots \times \mathbb{Z}^\mathbb{Z}$ by

$$\mu(A_1 \times A_2 \times \cdots \times A_d) = \prod_{l=1}^{d} \Big(\sum_{i=-\infty}^{\infty} \pi_i^{(l)} \tilde{P}_i^{(l)}(A_l) \Big),$$

and observe that μ is invariant under the $\mathbb{Z}^d$-action $\{\phi_{(i_1, i_2, \ldots, i_d)}\}$ on S defined as follows

$$\phi_{(i_1, i_2, \ldots, i_d)}((a_u^{(1)}), (a_u^{(2)}), \ldots, (a_u^{(d)})) = ((a_{u+i_1}^{(1)}), (a_{u+i_2}^{(2)}), \ldots, (a_{u+i_d}^{(d)})) .$$

This $\mathbb{Z}^d$-action is conservative (see Harris and Robbins (1953)).

Let $\mathbf{X} = \{X_{(i_1, i_2, \ldots, i_d)}\}$ be a stationary $S\alpha S$ random field defined by the integral representation (2.3.1) with M being a $S\alpha S$ random measure on S with control measure μ, and

$$f_{(i_1, i_2, \ldots, i_d)} = f \circ \phi_{(i_1, i_2, \ldots, i_d)} , \quad i_1, i_2, \ldots, i_d \in \mathbb{Z}$$

with

$$f(x^{(1)}, x^{(2)}, \dots, x^{(d)}) = I_{\{x_0^{(1)}=0,\, x_0^{(2)}=0,\, \dots,\, x_0^{(d)}=0\}}\,, \quad x^{(1)}, x^{(2)}, \dots, x^{(d)} \in \mathbb{Z}^{\mathbb{Z}}\,.$$

Assume further that for all $l = 1, 2, \dots, d$,

$$P_0(x_n^{(l)} = 0) \sim n^{-\gamma_l} L_l(n) \quad \text{as } n \to \infty$$

for some $\gamma_1,\ \gamma_2, \dots, \gamma_d \in (0, 1)$ such that $\gamma_1 + \gamma_2 + \cdots + \gamma_d > 1$ and slowly varying functions $L_1,\ L_2, \dots, L_d$. Using the calculations of Example 5.3 of Samorodnitsky (2004a) we have

$$\begin{aligned} b_n^\alpha &= \prod_{l=1}^{d} \Big(\sum_{i=-\infty}^{\infty} \pi_i^{(l)} P_i^{(l)}(x_k^{(l)} = 0 \text{ for some } k = 0, 1, \dots, n-1) \Big) \\ &\sim c(\gamma_1,\ \gamma_2, \dots, \gamma_d) \frac{n^{\gamma_1+\gamma_2+\cdots+\gamma_d}}{L_1(n)L_2(n)\dots L_d(n)} \end{aligned}$$

as $n \to \infty$ where $c(\gamma_1,\ \gamma_2, \dots, \gamma_d) = \prod_{l=1}^{d} \left(\Gamma(1+\gamma_l)\Gamma(1-\gamma_l)\right)^{-1}$. Hence it follows from Theorem 2.6.1 and Remark 2.6.11 that

$$\frac{L_1(n)^{1/\alpha} L_2(n)^{1/\alpha} \dots L_d(n)^{1/\alpha}}{n^{(\gamma_1+\gamma_2+\cdots+\gamma_d)/\alpha}} M_n \Rightarrow \left(C_\alpha\, c(\gamma_1,\ \gamma_2, \dots, \gamma_d)\right)^{1/\alpha} Z_\alpha$$

as $n \to \infty$.

Next is an example of an application of Theorem 2.7.4.

Example 2.8.4. Suppose $d = 3$, and define the $\mathbb{Z}^3$-action $\{\phi_{(i,j,k)}\}$ on $S = \mathbb{R} \times \{-1, 1\}$ as

$$\phi_{(i,j,k)}(x, y) = (x + i + 2j, (-1)^k y)\,.$$

An action-invariant measure μ on S is defined as the product of the Lebesgue measure on $\mathbb{R}$ and the counting measure on $\{-1, 1\}$. Take any $f \in L^\alpha(S)$ and define a stationary $S\alpha S$ random field $\{X_{(i,j,k)}\}$ as follows

$$X_{(i,j,k)} = \int_{\mathbb{R}\times\{-1,1\}} f(\phi_{(i,j,k)}(x, y))\, dM(x, y)\,,$$

where M is a $S\alpha S$ random measure on $\mathbb{R}\times\{-1,1\}$ with control measure μ. Note that the above representation of $\{X_{(i,j,k)}\}$ is of the form (2.3.3) generated by a measure preserving conservative action with $c_{(i,j,k)} \equiv 1$.

In the notation of Section 2.7 we have

$$K = \{(i, j, k) \in \mathbb{Z}^3 : \ i + 2j = 0 \text{ and } k \text{ is even}\},$$

and so

$$A \simeq \mathbb{Z}^3/K \simeq \mathbb{Z} \times \mathbb{Z}/2\mathbb{Z},$$

and

$$F = \{(i, 0, 0) : i \in \mathbb{Z}\}.$$

In particular $p = 1$ and $\{\phi_t\}_{t\in F}$ is dissipative. Hence Theorem 2.7.4 applies and says that $\frac{1}{n^{1/\alpha}} M_n$ converges to an extreme value distribution.

In all the examples we have seen so far, the action has a conservative direction i.e there is $u \in \mathbb{Z}^d - \{0\}$ such that $\{\phi_{nu}\}_{n\in\mathbb{Z}}$ is a conservative $\mathbb{Z}$-action. The following example of a $\mathbb{Z}^2$-action, suggested to us by M.G. Nadkarni, lacks such a conservative direction. In a sense, this example is “less one-dimensional” than the previous examples.

Example 2.8.5. Suppose that $d = 2$, and define the action $\{\phi_{(i,j)}\}_{i,j\in\mathbb{Z}}$ of $\mathbb{Z}^2$ on $S = \mathbb{R}$ with $\mu = Leb$ by

$$\phi_{(i,j)}(x) = x + i + j\sqrt{2}, \quad \text{for all } x \in \mathbb{R}.$$

Clearly, this action is measure preserving and it does not have any conservative direction. It is, however, well known that this action does not admit a wandering set of positive Lebesgue measure, and hence is conservative. In fact, if we take the kernel $f = I_{[0,1]}$ and define $\{X_{(i,j)}\}$ by (2.3.1) and (2.3.3) with, say, $c_{(i,j)} \equiv 1$, then we have for all $n \geq 2$,

$$b_n^\alpha = \mu\Big(\bigcup_{0\leq i,j\leq(n-1)} \phi_{(i,j)}([0,1])\Big) = \mu([0, 1 + (n-1)(1 + \sqrt{2})]).$$

So, $b_n \sim (1+\sqrt{2})^{1/\alpha} n^{1/\alpha}$ and a simple calculation shows that left hand side of (2.6.9) is bounded from above by $b_n^{-2\alpha}(\mu \otimes \mu)(B_n)$ where

$$B_n = \left\{(x,y) \in \mathbb{R}^2 : -(n-1)(1+\sqrt{2}) \leq x, y \leq 1, \, |x-y| \leq 1\right\}.$$

Since $(\mu \otimes \mu)(B_n) = O(n)$, (2.6.9) holds and hence

$$\frac{1}{n^{1/\alpha}} M_n \Rightarrow \left((1+\sqrt{2})C_\alpha\right)^{1/\alpha} Z_\alpha .$$

Chapter 3

Associated Point Processes

3.1 Introduction

Suppose, as in the previous chapter, that $\mathbf{X} := \{X_t\}_{t\in\mathbb{Z}^d}$ is a stationary $S\alpha S$ discrete-parameter random field. We consider the sequence of point processes on $[-\infty,\infty]-\{0\}$

$$N_n = \sum_{\|t\|_\infty \le n} \delta_{b_n^{-1}X_t}\,, \qquad n = 1,2,3,\ldots \tag{3.1.1}$$

induced by the random field $\mathbf{X}$ with an aptly chosen sequence of scaling constants $b_n \uparrow \infty$. Here δ_x denotes the point mass at x. We are interested in the weak convergence of this point process sequence in the space $\mathcal{M}$ of Radon measures on $[-\infty,\infty]-\{0\}$ equipped with the vague topology. This is important in extreme value theory because a number of limit theorems for various functionals of $S\alpha S$ random fields can be obtained by continuous mapping arguments on the associated point process sequence. See, for example, Resnick (1987) for a background on point processes and their applications to extreme value theory.

If $\{X_t\}_{t\in\mathbb{Z}^d}$ is an iid collection of random variables with tails decaying like those of a symmetric α stable distribution (i.e., $P(|X_t|>x) \sim Cx^{-\alpha}$ as $x\to\infty$ for all $t\in\mathbb{Z}^d$ and for some positive constant C) then $\{b_n\}$ can be chosen as follows:

$$b_n = n^{d/\alpha}\,. \tag{3.1.2}$$

With the above choice, the sequence $\{N_n\}$ converges weakly in the space $\mathcal{M}$ to a Poisson random measure, whose intensity blows up near zero (this is the reason why we exclude zero from the state space). See, once again, Resnick (1987). The assumption of independence can be relaxed to weak or local dependence provided the random field is stationary and the marginal distribution has balanced regularly varying tails. In this case, use of the same normalizing sequence (3.1.2) is still appropriate and $\{N_n\}$ converges weakly to a cluster Poisson process. See Davis and Resnick (1985) and Davis and Hsing (1995).

When the dependence structure of the random field is not necessarily weak or local, finding a suitable scaling sequence and computation of the weak limit both become challenging. As in the one-dimensional case in Resnick and Samorodnitsky (2004), we will observe that for stable random fields the choice of $\{b_n\}$ depends on the heaviness of the tails of the marginal distributions as well as on the length of memory. Therefore, in the short memory case the choice (3.1.2) of normalizing constants is appropriate although in the long memory case it is not. Furthermore, the extreme observations may cluster so much due to long memory that one may need to normalize the sequence $\{N_n\}$ itself to ensure weak convergence.

In Section 3.2 we study point processes corresponding to dissipative actions, i.e., point processes based on mixed moving averages. Section 3.3 deals with the conservative case using the connections to group theory established in Section 2.7.

3.2 The Dissipative Case

Let $\mathbf{X}$ be a stationary $S\alpha S$ discrete parameter random field generated by a dissipative $\mathbb{Z}^d$-action. We know from the previous chapter that such a random field has a mixed moving average representation (2.3.7) and its partial maxima sequence (2.5.1) grows exactly at the rate $n^{d/\alpha}$. As expected, $b_n \sim n^{d/\alpha}$ turns out to be the right normalization for the point process (3.1.1) in this case. The following theorem, which is a direct extension of Theorem 3.1 in Resnick and Samorodnitsky (2004), states that the limiting random measure is a cluster Poisson random measure even though the dependence structure is no longer weak or local.

Suppose l is the counting measure on $\mathbb{Z}^d$, ν is a σ-finite measure on a standard Borel space $(W, \mathcal{W})$ used in (2.3.7) and ν_α is the symmetric measure on $[-\infty, \infty] - \{0\}$ given by $\nu_\alpha(x, \infty] = \nu_\alpha[-\infty, -x) = x^{-\alpha}, \quad x > 0$. Without loss of generality, we assume that the original stable random field is of the form given in (2.3.7). Let

$$N = \sum_i \delta_{(j_i, v_i, u_i)} \sim PRM(\nu_\alpha \otimes \nu \otimes l) \tag{3.2.1}$$

be Poisson random measure on $([-\infty, \infty] - \{0\}) \times W \times \mathbb{Z}^d$ with mean measure $\nu_\alpha \otimes \nu \otimes l$. Then from the assumption above it follows that $\mathbf{X}$ has the following series representation:

$$X_t = C_\alpha^{\ 1/\alpha} \sum_i j_i f(v_i, u_i + t), \quad t \in \mathbb{Z}^d\,, \tag{3.2.2}$$

where C_α is the stable tail constant (2.6.2); see, for example, Samorodnitsky and Taqqu (1994).

Theorem 3.2.1. *Let* $\mathbf{X}$ *be the mixed moving average* (2.3.7)*, and define the point process* $N_n = \sum_{-n\mathbf{1} \le t \le n\mathbf{1}} \delta_{(2n)^{-d/\alpha} X_t}$, $n = 1, 2, \ldots$ *. Then* $N_n \Rightarrow N_*$ *as* $n \to \infty$*, weakly in the space* $\mathcal{M}$*, where* N_* *is a cluster Poisson random measure with representation*

$$N_* = \sum_{i=1}^{\infty} \sum_{t \in \mathbb{Z}^d} \delta_{j_i f(v_i, t)}\,, \tag{3.2.3}$$

where j_i*,* v_i *are described before* (3.2.1)*. Furthermore,* N_* *is Radon on* $[-\infty, \infty] - \{0\}$ *with Laplace functional (*$g \ge 0$ *continuous with compact support)*

$$\psi_{N_*}(g) = E\left(e^{-N_*(g)}\right) \tag{3.2.4}$$

$$= \exp\left\{ - \iint_{([-\infty,\infty]-\{0\})\times W} \left(1 - e^{-\sum_{t\in\mathbb{Z}^d} g\left(x f(v,t)\right)}\right) \nu_\alpha(dx)\nu(dv) \right\}.$$

Proof. The proof of this theorem is exactly same as the proof of Theorem 3.1 in Resnick and Samorodnitsky (2004). We first compute the Laplace functional of N_* in a similar fashion, namely, by observing that

$$E\left(e^{-N_*(g)}\right) = E\exp\left\{-\sum_i\sum_t g(j_i f(v_i,t))\right\} = E\exp\left\{-\sum_i \eta(j_i,v_i)\right\}$$

where

$$\eta(x,v) = \sum_{t\in\mathbb{Z}^d} g(xf(v,t))\,. \tag{3.2.5}$$

Since $\sum_i \delta_{(j_i,v_i)}$ is PRM with mean measure $\nu_\alpha \otimes \nu$, we have

$$E\left(e^{-N_*(g)}\right) = \exp\left\{-\iint_{([-\infty,\infty]-\{0\})\times W}(1-e^{-\eta(x,v)})\nu_\alpha(dx)\nu(dv)\right\}$$

which establishes (3.2.4).

To prove N_* is Radon, it is enough to show

$$E(N_*(h)) < \infty\,.$$

with $h(x) = 1_{[-\infty,-\delta]\cup[\delta,\infty]}$ and $\delta > 0$. Note that

$$E(N_*(h)) = E\sum_i\sum_{t\in\mathbb{Z}^d} h(j_i f(v_i,t)) = \sum_i E\eta'(j_i,v_i),$$

where we define η' by replacing g by h in (3.2.5). It follows from above that

$$\begin{aligned}
E(N_*(h)) &= \iint \eta'(x,v)\nu_\alpha(dx)\nu(dv) = \iint \sum_{t\in\mathbb{Z}^d} h(xf(v,t))\nu_\alpha(dx)\nu(dv)\\
&= \sum_{t\in\mathbb{Z}^d}\int_{v\in W}\left[\int_{|x|>\delta/|f(v,t)|}\nu_\alpha(dx)\right]\nu(dv)\\
&= 2\delta^{-\alpha}\sum_{t\in\mathbb{Z}^d}\int_W |f(v,t)|^\alpha \nu(dv) < \infty
\end{aligned}$$

since $f \in L^\alpha(W\times\mathbb{Z}^d, \nu\otimes l)$.

As in the one-dimensional case in Resnick and Samorodnitsky (2004) one can argue that only a few of j_i's in (3.2.2) are not nullified by the normalization $(2n)^{-d/\alpha}$ and hence we should expect to see a cluster Poisson process in the limit. Therefore, N_n should have the same weak limit as

$$N_n^{(2)} := \sum_{i=1}^{\infty} \sum_{\|t\|_\infty \le n} \delta_{(2n)^{-d/\alpha} j_i f(v_i, u_i + t)} \tag{3.2.6}$$

as $n \to \infty$. Our plan is to establish the convergence of $N_n^{(2)}$, and then show that N_n and $N_n^{(2)}$ converge to the same limit.

Using the method used to compute the Laplace functional of N_* and the scaling property of ν_α we get

$$\begin{aligned} & E\left(e^{-N_n^{(2)}(g)}\right) \qquad\qquad (3.2.7)\\ &= \exp\left\{-\frac{1}{(2n)^d}\int_{|x|>0}\int_W \sum_{u\in\mathbb{Z}^d}\left(1-e^{-\sum_{\|t\|_\infty\le n} g\left(xf(v,u+t)\right)}\right)\nu(dv)\nu_\alpha(dx)\right\} \end{aligned}$$

($g \ge 0$ continuous with compact support) which needs to be shown to converge to (3.2.4). As in the proof of (2.5.7) assume first that the function f in (2.3.7) is compactly supported, i.e., for some positive integer K

$$f(v,s) I_{W\times[-K\mathbf{1},K\mathbf{1}]^c}(v,s) \equiv 0\,. \tag{3.2.8}$$

Under this compact support assumption the integral in (3.2.7) equals

$$\begin{aligned} &\frac{1}{(2n)^d}\int_{|x|>0}\int_W \sum_{\|u\|_\infty\le n+K}\left(1-e^{-\sum_{\|t\|_\infty\le n} g\left(xf(v,u+t)\right)}\right)\nu(dv)\nu_\alpha(dx)\\ &= \frac{1}{(2n)^d}\int_{|x|>0}\int_W \sum_{u\in A_n}\left(1-e^{-\sum_{\|t\|_\infty\le n} g\left(xf(v,u+t)\right)}\right)\nu(dv)\nu_\alpha(dx)\\ &\quad + \frac{1}{(2n)^d}\int_{|x|>0}\int_W \sum_{\|u\|_\infty\le n-K}\left(1-e^{-\sum_{\|t\|_\infty\le n} g\left(xf(v,u+t)\right)}\right)\\ &\qquad\qquad\qquad\qquad\qquad\qquad \nu(dv)\nu_\alpha(dx)\\ &=: I_n + J_n \end{aligned}$$

for all $n > K$. Here $A_n = [-(n+K)\mathbf{1}, (n+K)\mathbf{1}] - [-(n-K)\mathbf{1}, (n-K)\mathbf{1}]$. We examine both integrals above and claim

$$I_n \to 0\,, \tag{3.2.9}$$

and

$$J_n \to \int_{|x|>0} \int_W \left(1 - e^{-\sum_{\|t\|_\infty \le K} g\left(xf(v,t)\right)}\right)\nu(dv)\nu_\alpha(dx)\,. \tag{3.2.10}$$

as $n \to \infty$.

Using the inequality $1 - e^{-x} \le x$, $(x > 0)$, and then the compact support assumption (3.2.8) we bound the first integral as follows:

$$\begin{aligned}
I_n &\le \frac{1}{(2n)^d} \int_{|x|>0} \int_W \sum_{u \in A_n} \sum_{\|t\|_\infty \le n} g\big(xf(v,u+t)\big)\nu(dv)\nu_\alpha(dx) \\
&\le \frac{1}{(2n)^d} \int_{|x|>0} \int_W \sum_{u \in A_n} \sum_{\|\tau\|_\infty \le K} g\big(xf(v,\tau)\big)\nu(dv)\nu_\alpha(dx) \\
&\le \frac{|A_n|}{(2n)^d} \int_{|x|>0} \int_W \sum_{\|\tau\|_\infty \le K} g\big(xf(v,\tau)\big)\nu(dv)\nu_\alpha(dx)
\end{aligned}$$

and since $g \ge 0$ has compact support on $[-\infty,\infty] - \{0\}$, $g \le C I_{[-\infty,-\delta]\cup[\delta,\infty]}$ (where $C, \delta > 0$), which yields

$$\begin{aligned}
&\le C\,\frac{|A_n|}{(2n)^d} \int_W \int_{|x|>0} \sum_{\|\tau\|_\infty \le K} I(|x| \ge \delta/|f(v,\tau)|)\nu_\alpha(dx)\nu(dv) \\
&\le C\,\frac{|A_n|}{(2n)^d}(2K+1)^d \int_W \nu_\alpha\Big(|x| \ge \frac{\delta}{\sum_{\|\tau\|_\infty \le K} |f(v,\tau)|}\Big)\nu(dv) \\
&\le \frac{2C(2K+1)^d |A_n|}{\delta^\alpha (2n)^d} \int_W \sum_{\|\tau\|_\infty \le K} |f(v,\tau)|^\alpha \nu(dv) \to 0
\end{aligned}$$

since $|A_n| = o(n^d)$ and $f \in L^\alpha(W \times \mathbb{Z}^d, \nu \otimes l)$. This proves (3.2.9).

To establish (3.2.10) observe that

$$\begin{aligned}
&J_n \\
&= \frac{1}{(2n)^d} \sum_{\|u\|_\infty \le n-K} \int_{|x|>0} \int_W \left(1 - e^{-\sum_{\|t\|_\infty \le n} g\left(xf(v,u+t)\right)}\right)\nu(dv)\nu_\alpha(dx) \\
&= \frac{(2n-2K+1)^d}{(2n)^d} \int_{|x|>0} \int_W \left(1 - e^{-\sum_{\|t\|_\infty \le K} g\left(xf(v,t)\right)}\right)\nu(dv)\nu_\alpha(dx) \\
&\to \int_{|x|>0} \int_W \left(1 - e^{-\sum_{\|t\|_\infty \le K} g\left(xf(v,t)\right)}\right)\nu(dv)\nu_\alpha(dx)
\end{aligned}$$

as $n \to \infty$ proving (3.2.10), which together with (3.2.9) completes the proof that $N_n^{(2)}$ converges to N^* weakly in $\mathcal{M}$ for compactly supported f.

We now remove the assumption of compact support on the function f by an exact same argument as in the one-dimensional case. For a general $f \in L^\alpha(\nu \otimes l)$ define

$$f_K(v,u) = f(v,u) I(\|u\|_\infty \le K), \quad K \ge 1. \tag{3.2.11}$$

Clearly each f_K satisfies (3.2.8) and $f_K \to f$ in $L^\alpha(\nu \otimes l)$ as $K \to \infty$. Define

$$N_n^{(2,K)} = \sum_{i=1}^{\infty} \sum_{\|t\|_\infty \le n} \delta_{(2n)^{-d/\alpha} j_i f_K(v_i, u_i + t)} \ , \tag{3.2.12}$$

for $K, n \ge 1$, and

$$N_*^{(K)} = \sum_{i=1}^{\infty} \sum_{t \in \mathbb{Z}^d} \delta_{j_i f_K(v_i, t)} \ , \quad K \ge 1\,. \tag{3.2.13}$$

For every $K \ge 1$, $N_n^{(2,K)} \Rightarrow N_*^{(K)}$ weakly in the space $\mathcal{M}$ as $n \to \infty$ by the arguments given above. As in the one-dimensional case, we will show

$$N_*^{(K)} \Rightarrow N_* \quad \text{weakly in the space } \mathcal{M} \text{ as } K \to \infty \tag{3.2.14}$$

and

$$\lim_{K \to \infty} \limsup_{n \to \infty} P(|N_n^{(2,K)}(g) - N_n^{(2)}(g)| > \epsilon) = 0 \tag{3.2.15}$$

for all $\epsilon > 0$ and for every non-negative continuous function g with compact support on $[-\infty, \infty] - \{0\}$. This will establish that N_* is the weak limit of $N_n^{(2)}$ as $n \to \infty$.

Since N_* is Radon, for every Borel set A bounded away from the origin N_* has finitely many points in A. Also, the set of points of $N_*^{(K)}$ increases to that of N_* as $K \to \infty$. This proves $N_*^{(K)} \xrightarrow{a.s.} N_*$ in $\mathcal{M}$ as $K \to \infty$, which implies (3.2.14).

Assuming as before $g \le C I_{[-\infty, -\delta] \cup [\delta, \infty]}$ (where $C, \delta > 0$), we have

$$E|N_n^{(2,K)}(g) - N_n^{(2)}(g)|$$

$$
\begin{aligned}
&= E\left(\sum_{i=1}^{\infty}\sum_{\|t\|_\infty\le n} g((2n)^{-d/\alpha} j_i f(v_i, u_i+t)) I(\|u_i+t\|_\infty > K)\right)\\
&= \sum_{\|t\|_\infty\le n} E\left(\sum_{i=1}^{\infty} g((2n)^{-d/\alpha} j_i f(v_i, u_i+t)) I(\|u_i+t\|_\infty > K)\right)\\
&= \sum_{\|t\|_\infty\le n}\int_W\int_{|x|>0}\sum_{u\in\mathbb{Z}^d} g((2n)^{-d/\alpha} x f(v, u+t)) I\Big(\|u+t\|_\infty > K\Big)\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \nu_\alpha(dx)\nu(dv)\\
&= \sum_{\|t\|_\infty\le n}\int_W\int_{|x|>0}\sum_{u\in\mathbb{Z}^d} g((2n)^{-d/\alpha} x f(v, u)) I(\|u\|_\infty > K)\nu_\alpha(dx)\nu(dv)\\
&= \frac{1}{(2n)^d}\sum_{\|t\|_\infty\le n}\int_W\int_{|x|>0}\sum_{\|u\|_\infty>K} g(x f(v,u))\nu_\alpha(dx)\nu(dv)\\
&\le C\frac{(2n+1)^d}{(2n)^d}\int_W\int_{|x|>0}\sum_{\|u\|_\infty>K} I(|x|>\delta/|f(v,u)|)\nu_\alpha(dx)\nu(dv)\\
&\le \frac{2C}{\delta^\alpha}\int_W\sum_{\|u\|_\infty>K}|f(v,u)|^\alpha\nu(dv)
\end{aligned}
$$

from which (3.2.15) follows since $f \in L_\alpha(\nu\otimes l)$. This proves $N_n^{(2)} \Rightarrow N_*$ for any kernel f.

To complete the proof of the theorem, we need to prove that for all $\epsilon > 0$

$$P[\rho(N_n, N_n^{(2)}) > \epsilon] \to 0 \quad (n\to\infty)$$

where ρ is the vague metric on $\mathcal{M}$. Clearly, it is enough to prove that for every non-negative continuous function g with compact support on $[-\infty,\infty]-\{0\}$,

$$
\begin{aligned}
&P(|N_n(g) - N_n^{(2)}(g)| > \epsilon)\\
&\qquad = P\left(\left|\sum_{\|t\|\le n}\left(g\left(\frac{X_t}{(2n)^{d/\alpha}}\right) - \sum_{i=1}^{\infty} g\left(\frac{j_i f(v_i, u_i+t)}{(2n)^{d/\alpha}}\right)\right)\right| > \epsilon\right) \to 0 \qquad (3.2.16)
\end{aligned}
$$

as $n\to\infty$, which follows by the exact same argument used to prove (3.14) in Resnick and Samorodnitsky (2004). This completes the proof of this result. □

3.3 Point Processes and Group Theory

This section deals with the longer memory case, i.e., the random field is now generated by a conservative action. In this case, we know from Theorem 2.6.1 that the partial maxima sequence (2.5.1) of the random field grows at a rate slower than $n^{d/\alpha}$. Hence (3.1.2) is inappropriate in this case. In general, there may or may not exist a normalizing sequence $\{b_n\}$ that ensures weak convergence of $\{N_n\}$. See Resnick and Samorodnitsky (2004) for examples of both kinds in the $d = 1$ case.

We will work with a specific class of stable random fields generated by conservative actions for which the effective dimension $p \leq d$ is known. For this class of random fields, the point process $\{N_n\}$ will not converge weakly to a nontrivial limit for any choice of the scaling sequence. Even for an appropriate choice of $\{b_n\}$, the associated point process won't even be tight (see Remark 3.3.4 below) because of the clustering effect of extreme observations due to longer memory of the random field. Hence, in order to ensure weak convergence, we have to normalize the point process sequence $\{N_n\}$ in addition to using a normalizing sequence $\{b_n\}$ different from (3.1.2). This phenomenon was also observed in Example 4.2 of the one-dimensional case in Resnick and Samorodnitsky (2004).

Our tools here are group theoretic as in Section 2.7; we study the algebraic structure of $A := \{\phi_t : t \in \mathbb{Z}^d\}$, a group of invertible nonsingular transformations on (S, μ) and use some basic counting arguments to analyze the point process $\{N_n\}$. We need to recall some of the notations and terminologies used in Section 2.7. We have a group homomorphism

$$\Phi : \mathbb{Z}^d \to A$$

defined by $\Phi(t) = \phi_t$. Letting $K := Ker(\Phi) = \{t \in \mathbb{Z}^d : \phi_t = 1_S\}$ as before, we get

$A \simeq \mathbb{Z}^d/K$, and hence

$$A = \bar{F} \oplus \bar{N}$$

where $\bar{F}$ is a free abelian group of rank p and $\bar{N}$ is a finite group of size l. Let Ψ be as before so that $F = \Psi(\bar{F})$ is a free subgroup of $\mathbb{Z}^d$ of rank p. As in Section 2.7 we assume that

$$p = rank(F) \geq 1$$

in this section as well (see Remark 2.7.5). From the proof of (2.7.1) we get that the sum $F + K$ is direct and

$$\mathbb{Z}^d/G \simeq \bar{N}\,,$$

where $G = F \oplus K$. Recall also that $x_1 + G,\ x_2 + G,\ \ldots\ , x_l + G$ are all the cosets of G in $\mathbb{Z}^d$ and $H := \bigcup_{k=1}^{l}(x_k + F)$, which becomes a group isomorphic to $\mathbb{Z}^d/K$ under the operation $\oplus$ (addition modulo K) defined in Section 2.7. Let $rank(K) = q \geq 1$ (we can also allow $q = 0$ provided we follow the convention mentioned in Remark 3.3.2). Choose a basis $\{u_1, u_2, \ldots, u_p\}$ of F and a basis $\{v_1, v_2, \ldots, v_q\}$ of K. Let U be the $d \times p$ matrix with u_i as its i^{th} column and V be the $d \times q$ matrix with v_j as its j^{th} column.

From Theorem 2.7.4 we can guess that $b_n \sim n^{p/\alpha}$ is a legitimate choice of the scaling sequence provided $\{\phi_t\}_{t \in F}$ is dissipative because, in that case, p becomes the intrinsic dimension of the random field. If $\{\phi_t\}_{t \in F}$ is a dissipative group action then we get another dissipative H-action $\{\psi_u\}_{u \in H}$ defined by (2.7.12). In this case, if we further assume that the cocycle in (2.3.3) satisfies

$$c_t \equiv 1 \quad \text{for all } t \in K, \tag{3.3.1}$$

then it will follow that $\{c_u\}_{u \in H}$ is an H-cocycle for $\{\psi_u\}_{u \in H}$, i.e., for all $u_1, u_2 \in H$,

$$c_{u_1 \oplus u_2}(s) = c_{u_1}(s) c_{u_2}(\psi_{u_1}(s)) \ \text{ for } \mu\text{-a.a. } s \in S.$$

Hence the subfield $\{X_u\}_{u\in H}$ is H-stationary and is generated by the dissipative action $\{\psi_u\}_{u\in H}$. Hence, by Remark 2.4.2, there is a standard Borel space $(W, \mathcal{W})$ with a σ-finite measure ν on it such that

$$X_u \stackrel{d}{=} \int_{W\times H} h(w, u\oplus s)\, M'(dw, ds), \quad u \in H, \tag{3.3.2}$$

for some $h \in L^\alpha(W \times H, \nu \otimes \tau)$, where τ is the counting measure on H, and M' is a $S\alpha S$ random measure on $W \times H$ with control measure $\nu \otimes \tau$.

Once again, we may assume, without loss of generality, that the original subfield $\{X_u\}_{u\in H}$ is given in the form (3.3.2). Let

$$N' = \sum_i \delta_{(j_i, v_i, u_i)} \sim PRM(\nu_\alpha \otimes \nu \otimes \tau) \tag{3.3.3}$$

be Poisson random measure on $([-\infty, \infty] - \{0\}) \times W \times H$ with mean measure $\nu_\alpha \otimes \nu \otimes \tau$. The following series representation holds in parallel to (3.2.2):

$$X_u = C_\alpha^{1/\alpha} \sum_{i=1}^{\infty} j_i h(v_i, u_i \oplus u), \quad u \in H, \tag{3.3.4}$$

where C_α is the stable tail constant (2.6.2) as before.

Define

$$C = \{y \in \mathbb{R}^p : \text{ there exists } \lambda \in \mathbb{R}^q \text{ such that } \|Uy + V\lambda\|_\infty \leq 1\}.$$

Let $|C|$ denote the p-dimensional volume of C, and for $y \in C$ denote by $\mathcal{V}(y)$ the q-dimensional volume of the polytope

$$P_y := \{\lambda \in \mathbb{R}^q : \|Uy + V\lambda\|_\infty \leq 1\}.$$

Define, for $t \in H$,

$$m(t, n) := \left|[-n\mathbf{1}, n\mathbf{1}] \cap (t + K)\right|. \tag{3.3.5}$$

Here $|B|$ denotes the cardinality of the finite set B. The following result, which is an extension of Theorem 3.2.1 (see Remark 3.3.2 below), states that the weak limit of properly scaled $\{N_n\}$ is a random measure which is not a point process.

Theorem 3.3.1. *Suppose* $\{\phi_t\}_{t\in F}$ *be a dissipative group action and* (3.3.1) *holds. Let* $\tilde{N}_n = n^{-q}\sum_{t\in[-n\mathbf{1},n\mathbf{1}]}\delta_{(cn)^{-p/\alpha}X_t}$, $n = 1, 2, \ldots$ *where* $c = (l|C|)^{1/p}$. *Then* $\tilde{N}_n \Rightarrow \tilde{N}_*$ *weakly in* $\mathcal{M}$*, where* $\tilde{N}_*$ *is a random measure with the following representation*

$$\tilde{N}_* = \sum_{i=1}^{\infty}\sum_{u\in H}\mathcal{V}(\xi_i)\delta_{j_i h(v_i,u)}\,, \tag{3.3.6}$$

where $\{j_i\}$ *and* $\{v_i\}$ *are as in* (3.3.3)*,* $\{\xi_i\}$ *is a sequence of iid p-dimensional random vectors uniformly distributed in C independent of* $\{j_i\}$ *and* $\{v_i\}$*, and* $\mathcal{V}$ *is the continuous function defined on C as above. Furthermore,* $\tilde{N}_*$ *is Radon on* $[-\infty,\infty]-\{0\}$ *with Laplace functional (*$g \geq 0$ *continuous with compact support)*

$$\begin{aligned}\psi_{\tilde{N}_*}(g) &= E\left(e^{-\tilde{N}_*(g)}\right) \qquad (3.3.7)\\ &= \exp\left\{-\frac{1}{|C|}\int_C\int_{|x|>0}\int_W(1 - e^{-\mathcal{V}(y)\sum_{w\in H}g(xh(v,w))})\nu(dv)\nu_\alpha(dx)dy\right\}.\end{aligned}$$

Remark 3.3.2. In the above theorem we can also allow q to be equal to 0 provided we follow the convention $\mathbb{R}^0 = \{0\}$, which is assumed to have 0-dimensional volume equal to 1. With these conventions, Theorem 3.3.1 reduces to Theorem 3.2.1 when $q = 0$.

In order to prove Theorem 3.3.1 we need some basic facts about C, $\mathcal{V}(y)$ and $m(t,n)$ defined above. We summarize them in the following

Lemma 3.3.3. *With the notations introduced above, we have:*

(i) C is compact and convex.

(ii) $\mathcal{V}(y)$ *is a continuous function of y.*

(iii) For all $1 \leq k \leq l$*,*

$$m_{k,n}(y) := \frac{m\left(x_k + \sum_{i=1}^{p}[ny_i]u_i, n\right)}{n^q}, \quad n = 1, 2, \ldots$$

($y = (y_1,\ldots,y_p)$) *is uniformly bounded on C and converges (as n* $\to\infty$*) to* $\mathcal{V}(y)$ *for all* $y \in C$.

(iv) There is a constant $\kappa_0 > 0$ such that $m(t,n)/n^q \leq \kappa_0$ for all $t \in H$ and for all $n \geq 1$. Also,

$$\frac{1}{n^p} \sum_{u \in H_n} \frac{m(u,n)}{n^q} \to l \int_C \mathcal{V}(y) dy < \infty$$

as $n \to \infty$. Here H_n is as in (2.7.21).

Proof. (i) Let $W = [U : V]$ and $z = \begin{bmatrix} y \\ \lambda \end{bmatrix}$. Then C is a projection of the closed and convex set

$$P := \{z \in \mathbb{R}^{p+q} : ||Wz||_\infty \leq 1\}.$$

To complete the proof of part (i) it is enough to establish that P is bounded because, in that case, C becomes a projection of a compact and convex set. To this end note that the columns of W are independent over $\mathbb{Z}$ and hence over $\mathbb{Q}$ which means that there is a $(p+q) \times d$ matrix Z such that $ZW = I$, the identity matrix of order $p+q$. From the string of inequalities

$$||z||_\infty = ||ZWz||_\infty \leq ||Z||_\infty ||Wz||_\infty \leq ||Z||_\infty$$

the boundedness of P follows.

(ii) Take $\{y^{(n)}\} \subseteq C$ such that $y^{(n)} \to y$. Fixing an integer $m \geq 1$ we can get N large enough so that for all $n \geq N$, $||y^{(n)} - y|| \leq \frac{1}{m}$ and hence

$$\begin{aligned} &\left\{\lambda \in \mathbb{R}^q : ||Uy + V\lambda||_\infty \leq 1 - \frac{||U||_\infty}{m}\right\} \\ &\subseteq P_{y^{(n)}} \subseteq \left\{\lambda \in \mathbb{R}^q : ||Uy + V\lambda||_\infty \leq 1 + \frac{||U||_\infty}{m}\right\}. \end{aligned}$$

First taking the lim sup (and lim inf) as $n \to \infty$ and then taking the limit as $m \to \infty$ we get that

$$\mathcal{V}(y) \leq \liminf_{n\to\infty} \mathcal{V}(y^{(n)}) \leq \limsup_{n\to\infty} \mathcal{V}(y^{(n)}) \leq \mathcal{V}(y)$$

which proves part (ii).

(iii) Fix $1 \le k \le l$. Let $L = \max_{1\le k\le l} \|x_k\|_\infty$. We start by showing that for all $y \in C$

$$m_{k,n}(y) \to \mathcal{V}(y) \tag{3.3.8}$$

as $n \to \infty$. Let

$$B_n := \Big\{ v \in \mathbb{Z}^q : \Big\| x_k + \sum_{i=1}^{p} [ny_i] u_i + Vv \Big\|_\infty \le n \Big\}, \quad n \ge 1 .$$

Since the columns of V are linearly independent over $\mathbb{Z}$, we have

$$|B_n| = \Big| [-n\mathbf{1}, n\mathbf{1}] \cap (x_k + \sum_{i=1}^{p} [ny_i] u_i + K) \Big| = n^q \, m_{k,n}(y) . \tag{3.3.9}$$

Define

$$C_m := \Big\{ \lambda \in \mathbb{R}^q : \|Uy + V\lambda\|_\infty \le 1 - \frac{1}{m} (\sum_{i=1}^{p} \|u_i\|_\infty + L) \Big\}, \quad m \ge 1 .$$

We first fix $m \ge 1$ and claim that for all $n \ge m$

$$\mathbb{Z}^q \cap nC_m \subseteq \mathbb{Z}^q \cap nC_n \subseteq B_n . \tag{3.3.10}$$

The first inequality is obvious. To prove the second one take

$$v \in \mathbb{Z}^q \cap nC_n = \Big\{ v \in \mathbb{Z}^q : \Big\| \sum_{i=1}^{p} ny_i \, u_i + Vv \Big\|_\infty \le n - \sum_{i=1}^{p} \|u_i\|_\infty - L \Big\}$$

and observe that

$$\begin{aligned} & \Big\| x_k + \sum_{i=1}^{p} [ny_i] u_i + Vv \Big\|_\infty \\ \le \quad & \|x_k\|_\infty + \Big\| \sum_{i=1}^{p} ny_i \, u_i + Vv \Big\|_\infty + \sum_{i=1}^{p} \|u_i\|_\infty \le n . \end{aligned}$$

It follows from (3.3.10) that

$$\frac{|\mathbb{Z}^q \cap nC_m|}{n^q} \le \frac{|B_n|}{n^q} = m_{k,n}(y) \tag{3.3.11}$$

for all $n \geq m$. Since C_m is a rational polytope (i.e., a polytope whose vertices have rational coordinates) the left hand side of (3.3.11) converges to $Volume(C_m)$, the q-dimensional volume of C_m by Theorem 1 of De Loera (2005). Hence (3.3.11) and (3.3.9) yield

$$Volume(C_m) \leq \liminf_{n\to\infty} m_{k,n}(y) .$$

Now taking another limit as $m \to \infty$ we get

$$\mathcal{V}(y) \leq \liminf_{n\to\infty} m_{k,n}(y) \tag{3.3.12}$$

since $C_m \uparrow P_y$. Defining another sequence of rational polytopes

$$C'_m := \left\{\lambda \in \mathbb{R}^q : \|Uy + V\lambda\|_\infty \leq 1 + \frac{1}{m}(\sum_{i=1}^{p} \|u_i\|_\infty + L)\right\}, \quad m \geq 1$$

and observing that $C'_m \downarrow P_y$ as $m \to \infty$ we can conclude using a similar argument

$$\limsup_{n\to\infty} m_{k,n}(y) \leq \mathcal{V}(y) . \tag{3.3.13}$$

(3.3.8) follows from (3.3.12) and (3.3.13).

To establish the uniform boundedness let $R := \sup_{y\in C} \|y\|_\infty < \infty$ by part (i). Observe that

$$C'_1 \subseteq \left\{\lambda \in \mathbb{R}^q : \|V\lambda\|_\infty \leq 1 + \sum_{i=1}^{p} \|u_i\|_\infty + L + R\|U\|_\infty\right\} =: C' ,$$

which is another rational polytope. Hence

$$m_{k,n}(y) \leq \frac{|\mathbb{Z}^q \cap nC'_1|}{n^q} \leq \frac{|\mathbb{Z}^q \cap nC'|}{n^q}$$

from which the uniform boundedness follows by another application of Theorem 1 of De Loera (2005).

(iv) To establish this part, we start by proving two set inclusions which will be useful once more later in this section. For $1 \leq k \leq l$, define

$$Q_n^{(k)} := \{\alpha \in \mathbb{Z}^p : \ x_k + U\alpha \in F_{k,n}\} , \quad n \geq 1.$$

Here $F_{k,n}$ is as in Lemma 2.7.3. Let $L = \max_{1\le k\le l} \|x_k\|_\infty$ and $L' = L + \sum_{i=1}^{p} \|u_i\|_\infty + \sum_{j=1}^{q} \|v_j\|_\infty$. We claim that for all $n > L'$,

$$\begin{aligned}
&\Big\{([(n-L')y_1], \ldots, [(n-L')y_p]) : \, y \in C\Big\} \\
&\quad \subseteq Q_n^{(k)} \subseteq \Big\{([(n+L)y_1], \ldots, [(n+L)y_p]) : \, y \in C\Big\} . \qquad (3.3.14)
\end{aligned}$$

To prove the first inclusion, let $y \in C$ and $\lambda \in \mathbb{R}^q$ be such that

$$\|Uy + V\lambda\|_\infty \le 1 .$$

Then we have

$$\begin{aligned}
&\left\| x_k + \sum_{i=1}^{p} [(n-L')y_i]u_i + \sum_{j=1}^{q} [(n-L')\lambda_j]v_j \right\|_\infty \\
&\le L + (n-L') \left\| \sum_{i=1}^{p} y_i u_i + \sum_{j=1}^{q} \lambda_j v_j \right\|_\infty + \sum_{i=1}^{p} \|u_i\|_\infty + \sum_{j=1}^{q} \|v_j\|_\infty \\
&\le n
\end{aligned}$$

proving $x_k + \sum_{i=1}^{p} [(n-L')y_i]u_i \in F_{k,n}$ and hence the first inclusion in (3.3.14). The second one is easy. If $\alpha \in Q_n^{(k)}$ then for some $\beta \in \mathbb{Z}^q$

$$\|x_k + U\alpha + V\beta\|_\infty \le n ,$$

and hence

$$\|U\alpha + V\beta\|_\infty \le n + L ,$$

which yields $y = (1/(n+L))\alpha \in C$ and establishes the second set inclusion in (3.3.14).

To prove the uniform boundedness in part (iv) we use (3.3.14) as follows:

$$\begin{aligned}
&\sup_{n\ge1} \sup_{t\in H} \frac{m(t,n)}{n^q} \\
&= \sup_{n\ge1} \max_{t\in H_n} \frac{m(t,n)}{n^q} \\
&\le \max_{1\le k\le l} \sup_{n\ge1} \max_{\alpha\in Q_n^{(k)}} \frac{m(x_k + U\alpha, n+L)}{n^q} \\
&\le \max_{1\le k\le l} \sup_{n\ge1} \sup_{y\in C} \left(1 + \frac{L}{n}\right)^q \frac{m(x_k + \sum_{i=1}^{p} [(n+L)y_i]u_i, n+L)}{(n+L)^q} ,
\end{aligned}$$

and this is bounded above by

$$\kappa_0 = (1+L)^q \max_{1\le k\le l} \sup_{n\ge 1} \sup_{y\in C} m_{k,n}(y)\,, \tag{3.3.15}$$

which is finite by part (iii).

To prove the convergence it is enough to show that for all $1 \le k \le l$

$$\frac{1}{n^p} \sum_{u\in F_{k,n}} \frac{m(u,n)}{n^q} \to \int_C \mathcal{V}(y)dy \quad (n\to\infty)\,. \tag{3.3.16}$$

To prove (3.3.16) we use (3.3.14) once again to get the following bound:

$$\begin{aligned}
&\frac{1}{n^p} \sum_{u\in F_{k,n}} \frac{m(u,n)}{n^q} \\
&\le \left(\frac{n+L}{n}\right)^p \frac{1}{(n+L)^p} \sum_{\alpha\in Q_n^{(k)}} \frac{m(x_k + U\alpha, n+L)}{n^q} \\
&\le \left(\frac{n+L}{n}\right)^{p+q} \int_C \frac{m\left(x_k + \sum_{i=1}^p [(n+L)y_i]u_i, n+L\right)}{(n+L)^q}\, dy + o(1)\,,
\end{aligned}$$

from which using part (iii) and dominated convergence theorem we get

$$\limsup_{n\to\infty} \frac{1}{n^p} \sum_{u\in F_{k,n}} \frac{m(u,n)}{n^q} \le \int_C \mathcal{V}(y)dy\,.$$

Similarly we can also prove

$$\liminf_{n\to\infty} \frac{1}{n^p} \sum_{u\in F_{k,n}} \frac{m(u,n)}{n^q} \ge \int_C \mathcal{V}(y)dy\,.$$

(3.3.16) follows from the above two inequalities. This completes the proof of Lemma 3.3.3. □

Proof of Theorem 3.3.1. The steps of this proof are similar to the proof of Theorem 3.2.1. We start with the Laplace functional of $\tilde{N}_*$,

$$\begin{aligned}
\psi_{\tilde{N}_*}(g) &= E\left(e^{-\tilde{N}_*(g)}\right) \\
&= E\exp\left\{-\sum_{i=1}^{\infty} \sum_{u\in H} \mathcal{V}(\xi_i) g(j_i h(v_i,u))\right\}
\end{aligned}$$

which equals (3.3.7) by a similar argument as before since $\sum_i \delta_{(j_i, v_i, \xi_i)}$ is PRM with mean measure $(1/|C|)(\nu_\alpha \otimes \nu \otimes Leb|_C)$.

To prove that $\tilde{N}_*$ is Radon we take $\eta(x) = 1_{[-\infty,-\delta]\cup[\delta,\infty]}$, $\delta > 0$ and look at

$$E\left(\tilde{N}_*(\eta)\right) = E\sum_{i=1}^{\infty}\sum_{u\in H}\mathcal{V}(\xi_i)\eta(j_i h(v_i,u)) \leq \|\mathcal{V}\|_\infty\, E\sum_{i=1}^{\infty}\sum_{u\in H}\eta(j_i h(v_i,u)),$$

where $\|\mathcal{V}\|_\infty := \sup_{y\in C}\mathcal{V}(y) < \infty$ by Lemma 3.3.3. From above, we get $E\left(\tilde{N}_*(\eta)\right) < \infty$ by an exact same argument as in the case of N_*. This proves $\tilde{N}_*$ is Radon.

Observe that because of (3.3.1) $\tilde{N}_n$ can also be written as

$$\tilde{N}_n = \sum_{t\in H_n}\frac{m(t,n)}{n^q}\,\delta_{(cn)^{-p/\alpha}X_t} \tag{3.3.17}$$

where $m(t,n)$ is as in (3.3.5) and H_n is as in (2.7.21). The weak convergence of $\tilde{N}_n$ is established in two steps as in the proof of the weak convergence of N_n. Namely, we first show that

$$\tilde{N}_n^{(2)} := \sum_{i=1}^{\infty}\sum_{t\in H_n}\frac{m(t,n)}{n^q}\,\delta_{(cn)^{-p/\alpha}j_i h(v_i,u_i\oplus t)} \tag{3.3.18}$$

converges to $\tilde{N}_*$ weakly in $\mathcal{M}$ and then show that $\tilde{N}_n$ must have the same weak limit as $\tilde{N}_n^{(2)}$.

Another use of the scaling property of ν_α yields the Laplace functional of $\tilde{N}_n^{(2)}$ ($g \geq 0$ continuous with compact support) as

$$\begin{aligned} &E\left(e^{-\tilde{N}_n^{(2)}(g)}\right) \\ &= \exp\left\{-\frac{1}{(cn)^p}\int_{|x|>0}\int_W\sum_{u\in H}\left(1-e^{-\frac{1}{n^q}\sum_{t\in H_n}m(t,n)\,g(xh(v,u\oplus t))}\right)\right. \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \left.\nu(dv)\nu_\alpha(dx)\right\} \end{aligned} \tag{3.3.19}$$

which needs to be shown to converge to (3.3.7). As before we first assume that h is compactly supported i.e., for some positive integer M

$$h(v,u)I_{W\times H_M^c}(v,u) \equiv 0\,. \tag{3.3.20}$$

Recall from Section 2.7 that each H_M is finite and $H_M \uparrow H$ as $M \to \infty$. Using properties (2.7.19), (2.7.20) and the compact support assumption (3.3.20) the integral in (3.3.19) becomes

$$\frac{1}{(cn)^p} \iint \sum_{u \in H_{n+M}} \left(1 - \exp\left(- \sum_{t \in H_n} \frac{m(t,n)}{n^q} g\big(xh(v, u \oplus t)\big)\right)\right) \nu(dv)\nu_\alpha(dx)\,,$$

which, by a change of variable, equals

$$\begin{aligned} &\frac{1}{(cn)^p} \iint \sum_{u \in H_{n+M}} \left(1 - \exp\left(- \sum_{w \in A'_n} \frac{m(w \ominus u, n)}{n^q} g\big(xh(v, w)\big)\right)\right) \nu(dv)\nu_\alpha(dx) \\ &=: \mathcal{I}_n\,. \end{aligned}$$

Here $w \ominus u := w \oplus u^{-1}$ and $A'_n = H_M \cap \{w' : w' \ominus u \in H_n\}$.

We claim that for all $n > M$

$$m(u^{-1}, n - M) \le m(w \ominus u, n) \le m(u^{-1}, n + M)\,. \tag{3.3.21}$$

The first inequality follows, for example, because

$$\tau \in [-(n-M)\mathbf{1}, (n-M)\mathbf{1}] \cap (u^{-1} + K)$$

if and only if

$$\tau + w \in [-n\mathbf{1}, n\mathbf{1}] \cap \big((w \ominus u) + K\big)\,.$$

Similarly we can prove the second inequality in (3.3.21).

We bound $\mathcal{I}_n$ using (3.3.21) by

$$\frac{1}{(cn)^p} \iint \sum_{u \in H_{n+M}} \left(1 - \exp\left(- \sum_{w \in A'_n} \frac{m(u^{-1}, n+M)}{n^q} g\big(xh(v, w)\big)\right)\right) \nu(dv)\nu_\alpha(dx)$$

$$= \frac{1}{(cn)^p} \iint \sum_{u \in H_{n+M}} \left(1 - \exp\left(- \sum_{w \in A'_n} \frac{m(u^{-1}, n)}{n^q} g\big(xh(v,w)\big)\right)\right) \nu(dv)\nu_\alpha(dx)$$

$$+ o(1) =: \mathcal{I}'_n + o(1)\,. \tag{3.3.22}$$

To prove (3.3.22) observe that using the inequality $|e^{-a} - e^{-b}| \leq |a - b|$, $(a, b > 0)$ the difference of the two integrals above can be bounded by

$$\frac{1}{(cn)^p} \sum_{u \in H_{n+M}} \left(\frac{m(u^{-1}, n+M) - m(u^{-1}, n)}{n^q}\right)$$
$$\times \iint \sum_{w \in H_M} g\big(xh(v,w)\big)\nu(dv)\nu_\alpha(dx)$$

which converges to 0 as $n \to \infty$ because

$$\iint \sum_{w \in H_M} g\big(xh(v,w)\big)\nu(dv)\nu_\alpha(dx) < \infty$$

by the exact same argument given in the proof of (3.2.9), and Lemma 3.3.3 together with (2.7.22) implies

$$\begin{aligned}
&\left|\frac{1}{(cn)^p} \sum_{u \in H_{n+M}} \left(\frac{m(u^{-1}, n+M) - m(u^{-1}, n)}{n^q}\right)\right| \\
=&\frac{1}{(cn)^p} \sum_{u \in H_{n+M}} \left(\left(\frac{n+M}{n}\right)^q \frac{m(u, n+M)}{(n+M)^q} - \frac{m(u,n)}{n^q}\right) \\
=&\, o(1) + \frac{1}{c^p}\Bigg[\left(\frac{n+M}{n}\right)^{p+q} \frac{1}{(n+M)^p} \sum_{u \in H_{n+M}} \frac{m(u, n+M)}{(n+M)^q} \\
&\qquad - \frac{1}{n^p} \sum_{u \in H_n} \frac{m(u,n)}{n^q}\Bigg] \to 0\,.
\end{aligned}$$

This proves (3.3.22), which yields $\mathcal{I}_n \leq \mathcal{I}'_n + o(1)$. Similarly we can also get a lower bound of $\mathcal{I}_n$ and establish that $\mathcal{I}_n \geq \mathcal{I}'_n + o(1)$. Hence, in order to complete the proof of weak convergence of $\tilde{N}_n^{(2)}$ to $\tilde{N}_*$ under the compact support assumption (3.3.20), it is enough to show that

$$\mathcal{I}'_n = \frac{1}{(cn)^p} \iint \sum_{u \in H_{n+M}} \left(1 - \exp\left(-\frac{m(u,n)}{n^q} \sum_{w \in A'_n} g\big(xh(v,w)\big)\right)\right) \nu(dv)\nu_\alpha(dx)$$

converges to

$$l\frac{1}{c^p}\int_C\int_{|x|>0}\int_W\left(1-\exp\left(-\mathcal{V}(y)\sum_{w\in H_M}g(xh(v,w))\right)\right)\nu(dv)\nu_\alpha(dx)dy\,. \tag{3.3.23}$$

To this end we decompose the integral $\mathcal{I}'_n$ into two parts as before.

$$\begin{aligned}
&\mathcal{I}'_n\\
&=\frac{1}{(cn)^p}\iint\sum_{u\in H_{n-M}}\left(1-\exp\left(-\frac{m(u,n)}{n^q}\sum_{w\in H_M}g\big(xh(v,w)\big)\right)\right)\nu(dv)\nu_\alpha(dx)\\
&\quad+\frac{1}{(cn)^p}\iint\sum_{u\in B'_n}\left(1-\exp\left(-\frac{m(u,n)}{n^q}\sum_{w\in A'_n}g\big(xh(v,w)\big)\right)\right)\nu(dv)\nu_\alpha(dx)\\
&=:J'_n+L'_n
\end{aligned}$$

for all $n>M$. Here $B'_n=H_{n+M}\cap H^c_{n-M}$. For $1\le k\le l$ let

$$\begin{aligned}
&J'_{k,n}\\
&=\frac{1}{(cn)^p}\iint\sum_{u\in F_{k,n-M}}\left(1-\exp\left(-\frac{m(u,n)}{n^q}\sum_{w\in H_M}g\big(xh(v,w)\big)\right)\right)\nu(dv)\nu_\alpha(dx)\,.
\end{aligned}$$

Clearly $J'_n=\sum_{k=1}^l J'_{k,n}$. We will show that each $J'_{k,n}$, $1\le k\le l$ converges to (3.3.23) except for the factor l.

Fix $k\in\{1,2,\dots,l\}$. Repeating the argument in the proof of (3.3.22) we obtain for all $n>M$,

$$\begin{aligned}
&J'_{k,n}\\
&=o(1)+\iint\frac{1}{(cn)^p}\sum_{u\in F_{k,n-M}}\left(1-e^{-\frac{m(u,n-M+L)}{n^q}\sum_{w\in H_M}g\left(xh(v,w)\right)}\right)\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\nu(dv)\nu_\alpha(dx)\\
&=o(1)\;+\left(\frac{n-M+L}{cn}\right)^p\times\\
&\quad\iint\frac{1}{(n-M+L)^p}\sum_{\alpha\in Q^{(k)}_{n-M}}\left(1-e^{-\frac{m(x_k+U\alpha,n-M+L)}{n^q}\sum_{w\in H_M}g\left(xh(v,w)\right)}\right)\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\nu(dv)\nu_\alpha(dx)
\end{aligned}$$

which can be estimated using (3.3.14) as follows:

$$\leq o(1) + \left(\frac{n-M+L}{cn}\right)^p \times$$
$$\int_{|x|>0}\int_W\int_C\left(1 - e^{-\frac{m(x_k+\sum_{i=1}^p[(n-M+L)y_i]u_i,n-M+L)}{n^q}\sum_{w\in H_M} g\left(xh(v,w)\right)}\right)$$
$$dy\,\nu(dv)\,\nu_\alpha(dx)\,.$$

By Lemma 3.3.3 there is a constant $\kappa > 0$ such that the above integrand sequence is dominated by

$$1 - \exp\left\{-\kappa\sum_{w\in H_M} g(xh(v,w))\right\}$$

which can be shown to be integrable using an argument similar to the one in the proof of (3.2.9). Hence Lemma 3.3.3 together with dominated convergence theorem yields

$$\int_{|x|>0}\int_W\int_C\left(1 - e^{-\frac{m(x_k+\sum_{i=1}^p[(n-M+L)y_i]u_i,n-M+L)}{n^q}\sum_{w\in H_M} g\left(xh(v,w)\right)}\right)$$
$$dy\,\nu(dv)\,\nu_\alpha(dx)$$
$$\to \int_C\int_{|x|>0}\int_W\left(1 - \exp\left(-\mathcal{V}(y)\sum_{w\in H_M} g(xh(v,w))\right)\right)\nu(dv)\nu_\alpha(dx)dy\,.$$

This shows

$$\limsup_{n\to\infty} J'_{k,n}$$
$$\leq \frac{1}{c^p}\int_C\int_{|x|>0}\int_W\left(1 - \exp\left(-\mathcal{V}(y)\sum_{w\in H_M} g(xh(v,w))\right)\right)\nu(dv)\nu_\alpha(dx)dy\,.$$

Similarly we can also prove that

$$\liminf_{n\to\infty} J'_{k,n}$$
$$\geq \frac{1}{c^p}\int_C\int_{|x|>0}\int_W\left(1 - \exp\left(-\mathcal{V}(y)\sum_{w\in H_M} g(xh(v,w))\right)\right)\nu(dv)\nu_\alpha(dx)dy\,.$$

Hence, J_n' converges to (3.3.23) as $n \to \infty$. To establish the weak convergence of $\tilde{N}_n^{(2)}$ when h is compactly supported it remains to prove that $L_n' \to 0$ as $n \to \infty$. This is easy because

$$\begin{aligned}
&L_n' \\
&\le \frac{1}{(cn)^p} \iint \sum_{u \in B_n'} \left(1 - \exp\left(-\frac{m(u,n)}{n^q} \sum_{w \in H_M} g\big(xh(v,w)\big)\right)\right) \nu(dv)\nu_\alpha(dx) \\
&= \frac{1}{(cn)^p} \iint \sum_{u \in H_{n+M}} \left(1 - \exp\left(-\frac{m(u,n)}{n^q} \sum_{w \in H_M} g\big(xh(v,w)\big)\right)\right) \\
&\qquad \nu(dv)\nu_\alpha(dx) \\
&- \frac{1}{(cn)^p} \iint \sum_{u \in H_{n-M}} \left(1 - \exp\left(-\frac{m(u,n)}{n^q} \sum_{w \in H_M} g\big(xh(v,w)\big)\right)\right) \\
&\qquad \nu(dv)\nu_\alpha(dx) \\
&\to 0
\end{aligned}$$

since the first term can also be shown to converge to the same limit as the second term by the exact same argument as above.

To remove the assumption of compact support on the function h, for a general $h \in L^\alpha(\nu \otimes \tau)$ define

$$h_M(v,u) = h(v,u) I_{H_M}(u), \quad M \ge 1. \tag{3.3.24}$$

Notice that each h_M satisfies (3.3.20) and that $h_M \to h$ almost surely as well as in $L^\alpha(\nu \otimes \tau)$ as $M \to \infty$. Denote

$$\tilde{N}_n^{(2,M)} = \sum_{i=1}^{\infty} \sum_{t \in H_n} \frac{m(t,n)}{n^q} \delta_{(cn)^{-p/\alpha} j_i h_M(v_i, u_i \oplus t)}, \tag{3.3.25}$$

for $M, n \ge 1$, and

$$\tilde{N}_*^{(M)} = \sum_{i=1}^{\infty} \sum_{u \in H} \mathcal{V}(\xi_i) \delta_{j_i h_M(v_i, u)}, \quad M \ge 1 \tag{3.3.26}$$

with the notations as above. We already know that for every $M \ge 1$, $\tilde{N}_n^{(2,M)} \Rightarrow \tilde{N}_*^{(M)}$ weakly in the space $\mathcal{M}$ as $n \to \infty$. Therefore, to establish $\tilde{N}_n^{(2)} \Rightarrow \tilde{N}_*$, it is enough to show two things:

$$\tilde{N}_*^{(M)} \Rightarrow \tilde{N}_* \quad \text{weakly as } M \to \infty \tag{3.3.27}$$

and

$$\lim_{M\to\infty} \limsup_{n\to\infty} P(|\tilde{N}_n^{(2,M)}(g) - \tilde{N}_n^{(2)}(g)| > \epsilon) = 0 \tag{3.3.28}$$

for all $\epsilon > 0$ and for every non-negative continuous function g with compact support on $[-\infty, \infty] - \{0\}$. Claim (3.3.27) is easy since the Laplace functional of $\tilde{N}_*^{(M)}$, which is obtained by replacing h in (3.3.7) by h_M, converges by dominated convergence theorem to (3.3.7) for every non-negative continuous function g with compact support on $[-\infty, \infty] - \{0\}$. The proof of (3.3.28) is along the same line as the proof of (3.2.15). Using similar calculations we have

$$\begin{aligned}
&E|\tilde{N}_n^{(2,M)}(g) - \tilde{N}_n^{(2)}(g)| \\
&= \sum_{t\in H_n} \frac{m(t,n)}{n^q} E\left(\sum_{i=1}^{\infty} g((cn)^{-p/\alpha} j_i h(v_i, u_i \oplus t)) I(N(u_i \oplus t) > M) \right) \\
&= \left(\frac{1}{(cn)^p} \sum_{t\in H_n} \frac{m(t,n)}{n^q} \right) \int_W \int_{|x|>0} \sum_{u\in H_M^c} g(xf(v,u))\nu_\alpha(dx)\nu(dv)\,.
\end{aligned}$$

The integral

$$\int_W \int_{|x|>0} \sum_{u\in H_M^c} g(xf(v,u))\nu_\alpha(dx)\nu(dv) \to 0$$

as $M \to \infty$ by repeating the argument given in the proof of (3.2.15). Hence, by Lemma 3.3.3, (3.3.28) follows and so does $\tilde{N}_n^{(2)} \Rightarrow \tilde{N}_*$ without the assumption of compact support.

To complete the proof of the theorem, we need to prove (with ρ being the vague metric on $\mathcal{M}$) that for all $\epsilon > 0$

$$P[\rho(\tilde{N}_n, \tilde{N}_n^{(2)}) > \epsilon] \to 0 \quad (n \to \infty)$$

and for this, it suffices to show that for every non-negative continuous function g with compact support on $[-\infty, \infty] - \{0\}$,

$$
\begin{aligned}
& P(|\tilde{N}_n(g) - \tilde{N}_n^{(2)}(g)| > \epsilon) \\
&= P\left(\left| \sum_{t \in H_n} \frac{m(t,n)}{n^q} \left(g\left(\frac{X_t}{(cn)^{p/\alpha}} \right) - \sum_{i=1}^{\infty} g\left(\frac{j_i h(v_i, u_i \oplus t)}{(cn)^{p/\alpha}} \right) \right) \right| > \epsilon \right) \qquad (3.3.29) \\
&\to 0
\end{aligned}
$$

as $n \to \infty$. By Lemma 3.3.3, (3.3.29) would follow from

$$
P\left(\left| \sum_{t \in H_n} \left(g\left(\frac{X_t}{(cn)^{p/\alpha}} \right) - \sum_{i=1}^{\infty} g\left(\frac{j_i h(v_i, u_i \oplus t)}{(cn)^{p/\alpha}} \right) \right) \right| > \epsilon/\kappa_0 \right) \to 0\,. \qquad (3.3.30)
$$

Here κ_0 is as in (3.3.15). Once again, following verbatim the proof of (3.14) in Resnick and Samorodnitsky (2004), we can establish (3.3.30) and complete the proof of this theorem. □

Remark 3.3.4. Note that the above theorem together with Lemma 3.20 in Resnick (1987) implies that the sequence of point process (3.1.1) with the choice $b_n \sim n^{p/\alpha}$ is not tight and hence does not converge weakly in $\mathcal{M}$. Furthermore, $\{N_n\}$ will not converge weakly to a nontrivial limit for any other choice of normalizing sequence $\{b_n\}$. All the points of $\{N_n\}$ will be driven to zero if b_n grows faster than $n^{p/\alpha}$. This follows from (2.7.23), which also implies that if we select b_n to grow slower than $n^{p/\alpha}$ then we will see an accumulation of mass at infinity. Only $b_n \sim n^{p/\alpha}$ places the points at the right scale, but they repeat so much due to long memory, that the point process itself has to be normalized by n^q in order to ensure weak convergence.

We end this section by considering a simple example and computing the weak limit of the corresponding random measure (properly normalized $\{N_n\}$) using Theorem 3.3.1. This will help us understand the result as well as get used to the notations.

Example 3.3.1. Suppose $d = 2$, and define the $\mathbb{Z}^2$-action $\{\phi_{(t_1,t_2)}\}$ on $S = \mathbb{R}$ as

$$
\phi_{(t_1,t_2)}(x) = x + t_1 - t_2\,.
$$

Take any $f \in L^\alpha(S,\mu)$ where μ is the Lebesgue measure on $\mathbb{R}$ and define a stationary $S\alpha S$ random field $\{X_{(t_1,t_2)}\}$ as follows

$$X_{(t_1,t_2)} = \int_{\mathbb{R}} f(\phi_{(t_1,t_2)}(x))\, M(dx), \quad t_1, t_2 \in \mathbb{Z}\,,$$

where M is a $S\alpha S$ random measure on $\mathbb{R}$ with control measure μ. Note that the above representation of $\{X_{(t_1,t_2)}\}$ is of the form (2.3.3) generated by a measure preserving conservative action with $c_{(t_1,t_2)} \equiv 1$.

In this case, using the notations as above, we have

$$K = \{(t_1,t_2) \in \mathbb{Z}^2 : \; t_1 = t_2\}$$

which implies $A \simeq \mathbb{Z}^2/K \simeq \mathbb{Z}$, and

$$F = \{(t_1, 0) : \; t_1 \in \mathbb{Z}\}\,.$$

In particular we have $p = q = l = 1$, and

$$U = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \; V = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

so that

$$\begin{aligned} C &= \{y \in \mathbb{R} : \text{ there exists } \lambda \in \mathbb{R} \text{ such that } \|Uy + V\lambda\|_\infty \leq 1\} \\ &= \{y \in \mathbb{R} : \; |y+\lambda| \leq 1 \text{ for some } \lambda \in [-1,1]\} \; = \; [-2,2]\,. \end{aligned}$$

For all $y \in C = [-2,2]$ we have

$$P_y = \{\lambda \in [-1,1] : \; |y+\lambda| \leq 1\} = \begin{cases} [-(1+y), 1] & y \in [-2,0) \\ [-1, 1-y] & y \in [0,2] \end{cases}$$

which yields

$$\mathcal{V}(y) = 2 - |y|\,, \quad y \in [-2,2]\,.$$

Clearly, $\{X_{(t_1,0)}\}_{t_1\in\mathbb{Z}}$ is a stationary $S\alpha S$ process generated by a dissipative flow $\{\phi_{(t_1,0)}\}_{t_1\in\mathbb{Z}}$. Hence, by Theorem 4.4 in Rosiński (1995), there is a σ-finite standard measure space (W,ν) and a function $h \in L^\alpha(W\times\mathbb{Z}, \nu\otimes l_{\mathbb{Z}})$ such that

$$X_{(t_1,0)} \stackrel{d}{=} \int_{W\times\mathbb{Z}} h(v, t_1+s)M(dv,ds)\,, \quad t_1 \in \mathbb{Z}\,.$$

Here $l_{\mathbb{Z}}$ is the counting measure on $\mathbb{Z}$, and M is a $S\alpha S$ random measure on $W\times\mathbb{Z}$ with control measure $\nu\otimes l_{\mathbb{Z}}$. Let

$$\sum_{i=1}^{\infty} \delta_{(j_i,v_i,\xi_i)} \sim PRM(\nu_\alpha \otimes \nu \otimes \frac{1}{4}Leb|_{[-2,2]})$$

be Poisson random measure on $([-\infty,\infty]-\{0\})\times W\times[-2,2]$. In this example, $c = (l|C|)^{1/p} = 4$ and

$$\tilde{N}_n = n^{-1}\sum_{|t_1|,\,|t_2|\le n} \delta_{(4n)^{-1/\alpha}X_{(t_1,t_2)}}\,, \quad n = 1,2,\ldots\,.$$

Since $\{\phi_u\}_{u\in F}$ is a dissipative group action and (3.3.1) holds in this case, we can use Theorem 3.3.1 and conclude that

$$\tilde{N}_n \Rightarrow \sum_{i=1}^{\infty}\sum_{t_1\in\mathbb{Z}} (2-|\xi_i|)\,\delta_{j_i h(v_i,t_1)}$$

weakly in the space $\mathcal{M}$.

Remark 3.3.5. Note that $\tilde{N}_n$ can also be written as follows:

$$\tilde{N}_n = \sum_{k=-2n}^{2n}\left(2-\frac{|k|}{n}+\frac{1}{n}\right)\delta_{(4n)^{-1/\alpha}Y_k}$$

where $Y_k = X_{(k,0)}$. Only a few of the Y_k's are not driven to zero by the normalization $b_n = (4n)^{-1/\alpha}$. By stationarity, these rare k's are distributed uniformly in $\{-2n,-2n+1,\ldots,2n\}$. Hence, one should expect the above weak limit of $\tilde{N}_n$.

Chapter 4

Continuous Parameter Fields

4.1 Introduction

In this chapter, we look at stationary $S\alpha S$ continuous parameter non-Gaussian random fields, which can be defined in parallel to the discrete-parameter case (see Section 2.1). Our goal is to present the continuous-parameter analogue of some of the results in Chapter 2. We will assume throughout this chapter that the random field $\mathbf{X} = \{X_t\}_{t\in\mathbb{R}^d}$ is measurable.

The connection to the ergodic theory of nonsingular group actions is present in the continuous-parameter case as well (see, for example, Rosiński (1995) and Rosiński (2000)). However, the notions of conservative and dissipative actions were not known for nonsingular $\mathbb{R}^d$-actions. In Section 4.2 we develop these notions and obtain a conservative-dissipative decomposition for the $\mathbb{R}^d$ case using a result of Varadarajan (1970). The major difference with the discrete-parameter case is the following: when we make statements about sets (e.g., equality or disjointness of two sets) which are understood as holding up to a set of measure zero with respect to the underlying measure, we had better be careful, or else we may end up with a measurability problem (see, for instance, Remark 4.2.3). This makes Section 4.2 more technical and challenging than Section 2.2.

Section 4.3 focuses on the continuous-parameter extensions of the structure results presented in Section 2.3. In particular, we connect the decomposition of $S\alpha S$ random fields obtained in Rosiński (2000) to the ergodic theoretic notions introduced in Section 4.3. We also observe, in this section, that any stationary measurable random field is continuous in probability, which is applied in Section 4.4 to get a separable version of $\mathbf{X}$ and avoid measurability problems related to an uncountable maximum.

The main objective of Section 4.4 is to compute the rate of growth of the maxima of the random field and establish a continuous-parameter analogue of the phase transition observed in Theorem 2.6.1. As in the discrete-parameter case, this phase transition can be regarded as a change from short to long memory of the field.

4.2 More Ergodic Theory

In this section we build up the theory of nonsingular $\mathbb{R}^d$-actions based on the results discussed in Section 2.2. Suppose $(S, \mathcal{S}, \mu)$ is a σ-finite standard measure space and $(G, +)$ is a topological group with identity element 0 and Borel σ-field $\mathcal{G}$. A collection of maps $\phi_t : S \to S,\ t \in G$ is called a group action of G on S if

1. $(t, s) \mapsto \phi_t(s)$ is jointly measurable,
2. ϕ_0 is the identity map on S, and
3. $\phi_{u+v} = \phi_u \circ \phi_v$ for all $u, v \in G$.

A group action $\{\phi_t\}_{t \in G}$ of G on S is called nonsingular if $\mu \circ \phi_t \sim \mu$ for all $t \in G$.

From Section 2.2 we know if G is countable then we can define conservative and dissipative parts of the group action $\{\phi_t\}_{t \in G}$ using the notion of wandering sets. It is impossible to do the same for an action of an uncountable group. However, the following machineries enable us to define conservative and dissipative parts for nonsingular $\mathbb{R}^d$-actions using the structure of $\mathbb{R}^d$. Let $\mathcal{B}$ be the Borel σ-field and λ be the Lebesgue measure on $\mathbb{R}^d$.

Definition 4.2.1. *A countable subgroup $\Gamma \subseteq \mathbb{R}^d$ is called a lattice in $\mathbb{R}^d$ if there exists $F \in \mathcal{B}$ with $\lambda(F) < \infty$ such that $\{\gamma + F : \gamma \in \Gamma\}$ are disjoint and $\bigcup_{\gamma \in \Gamma}(\gamma + F) = \mathbb{R}^d$.*

Clearly, $\Gamma_n := \frac{1}{2^n}\mathbb{Z}^d$ are lattices in $\mathbb{R}^d$ for all $n \geq 0$. This sequence of lattices will play a very significant role in the following result, which is a partial extension of Proposition 1.6.4 and Corollary 1.6.5 in Aaronson (1997) to not necessarily measure preserving $\mathbb{R}^d$-actions. This result enables us to define conservative and dissipative parts of a non-singular $\mathbb{R}^d$-action.

Proposition 4.2.1. *Conservative (resp. dissipative) parts of the actions $\{\phi_t\}_{t \in \Gamma_n}$, $n \geq 0$, are all equal modulo μ.*

Proof. For all $n \geq 0$, let C_n be the conservative part of $\{\phi_t\}_{t \in \Gamma_n}$. Fix $m \geq 0$. We will show

$$C_m = C_0 \mod \mu\,. \tag{4.2.1}$$

By Theorem 8.10 of Varadarajan (1970) there exists a jointly measurable real valued function $(t, s) \mapsto w_t(s)$ on $\mathbb{R}^d \times S$ which is positive everywhere and for λ-almost all $t \in \mathbb{R}^d$, $s \mapsto w_t(s)$ is a version of the Radon-Nikodym derivative $\frac{d\mu \circ \phi_t}{d\mu}$. Without loss of generality, we can assume that $w_\gamma(\cdot)$ is a version of $\frac{d\mu \circ \phi_\gamma}{d\mu}$ for all $\gamma \in \Gamma_m$ and $w_0 \equiv 1$. Let $F_m := [0, \frac{1}{2^m}\mathbf{1})$, where $\mathbf{1} = (1, 1, \ldots, 1) \in \mathbb{R}^d$ as before, but, for all $u = (u^{(1)}, u^{(2)}, \ldots, u^{(d)}) \in \mathbb{R}^d$, $v = (v^{(1)}, v^{(2)}, \ldots, v^{(d)}) \in \mathbb{R}^d$,

$$[u, v) := \{x \in \mathbb{R}^d : u^{(i)} \leq x^{(i)} \leq v^{(i)} \text{ for all } i = 1, 2, \ldots, d\}\,. \tag{4.2.2}$$

We define, for all $t \in F_m$, and for all $\gamma \in \Gamma_m$,

$$w^{(m)}_{\gamma+t}(s) = w_t \circ \phi_\gamma(s)\, w_\gamma(s)\,. \tag{4.2.3}$$

Clearly for λ-almost all $\tau \in \mathbb{R}^d$, the above definition yields $s \mapsto w^{(m)}_\tau(s)$ to be a version of the Radon-Nikodym derivative $\frac{d\mu \circ \phi_\tau}{d\mu}$ keeping $(\tau, s) \mapsto w^{(m)}_\tau(s)$ jointly measurable. Let

$$S_0 := \{s \in S : w_{\gamma_1+\gamma_2}(s) = w_{\gamma_1} \circ \phi_{\gamma_2}(s)\, w_{\gamma_2}(s) \text{ for all } \gamma_1, \gamma_2 \in \Gamma_m\}\,.$$

Then, by our assumption, $\mu(S - S_0) = 0$. We claim that for all $s_0 \in S_0$ we have

$$w^{(m)}_{\gamma_0+t_0}(s_0) = w^{(m)}_{t_0} \circ \phi_{\gamma_0}(s_0)\, w^{(m)}_{\gamma_0}(s_0) \quad \text{for all } \gamma_0 \in \Gamma_0,\ t_0 \in F_0\,. \tag{4.2.4}$$

To prove this, let $t_0 = \gamma_1 + t_1$ where $\gamma_1 \in \Gamma_m$ and $t_1 \in F_m$. Then $\gamma_1' = \gamma_0 + \gamma_1 \in \Gamma_m$. For all $s_0 \in S_0$, repeated use of (4.2.3) yields,

$$\begin{aligned}
w^{(m)}_{\gamma_0+t_0}(s_0) &= w^{(m)}_{\gamma_1'+t_1}(s_0)\\
&= w^{(m)}_{t_1} \circ \phi_{\gamma_1'}(s_0)\, w^{(m)}_{\gamma_1'}(s_0)\\
&= w^{(m)}_{t_1} \circ \phi_{\gamma_1'}(s_0)\, w^{(m)}_{\gamma_1} \circ \phi_{\gamma_0}(s_0)\, w^{(m)}_{\gamma_0}(s_0)\\
&= \left[\left(w^{(m)}_{t_1} \circ \phi_{\gamma_1}\right) w^{(m)}_{\gamma_1}\right] \circ \phi_{\gamma_0}(s_0)\, w^{(m)}_{\gamma_0}(s_0)\\
&= w^{(m)}_{t_0} \circ \phi_{\gamma_0}(s_0)\, w^{(m)}_{\gamma_0}(s_0)
\end{aligned}$$

proving (4.2.4).

Taking $g \in L^1(S, \mu)$, $g > 0$, we get, for all $s_0 \in S_0$,

$$\begin{aligned}
\int_{\mathbb{R}^d} h \circ \phi_t(s_0) w^{(m)}_t(s_0)\lambda(dt) &= \sum_{\gamma \in \Gamma_0} \int_{\gamma + F_0} h \circ \phi_t(s_0) w^{(m)}_t(s_0)\lambda(dt)\\
&= \sum_{\gamma \in \Gamma_0} \int_{F_0} h \circ \phi_{\gamma+t}(s_0) w^{(m)}_{\gamma+t}(s_0)\lambda(dt)\\
&= \sum_{\gamma \in \Gamma_0} g_0 \circ \phi_\gamma(s_0) w^{(m)}_\gamma(s_0)
\end{aligned}$$

by (4.2.4), where

$$g_0(s) := \int_{F_0} h \circ \phi_t(s) w^{(m)}_t(s)\lambda(dt) > 0\,, \quad s \in S\,.$$

Using Fubini's Theorem we have

$$\begin{aligned}
\int_S g_0(s)\mu(ds) &= \int_{F_0}\int_S h \circ \phi_t(s) w^{(m)}_t(s)\mu(ds)\lambda(dt)\\
&= \lambda(F_0)\int_S h(s)\mu(ds) < \infty\,,
\end{aligned}$$

which proves $g_0 \in L^1(S,\mu)$. Similarly we can show using (4.2.3) and Fubini's Theorem that for all $s \in S$,

$$\int_{\mathbb{R}^d} h \circ \phi_t(s) w_t^{(m)}(s) \lambda(dt) = \sum_{\gamma \in \Gamma_m} g_m \circ \phi_\gamma(s) w_\gamma^{(m)}(s),$$

where $g_m(s) := \int_{F_m} h \circ \phi_t(s) w_t^{(m)}(s) \lambda(dt) > 0$, $g_m \in L^1(S,\mu)$. Hence, by Corollary 2.2.4 we get

$$C_m \cap S_0 = C_0 \cap S_0 = \{s \in S_0 : \int_{\mathbb{R}^d} h \circ \phi_t(s) w_t^{(m)}(s) \lambda(dt) = \infty\}$$

from which (4.2.1) follows. $\square$

Motivated by this proposition, we introduce the notion of conservative and dissipative parts of an $\mathbb{R}^d$-action as follows.

Definition 4.2.2. *The conservative (resp. dissipative) part of* $\{\phi_t\}_{t \in \mathbb{R}^d}$ *is defined to be* C_0 *(resp.* $\mathcal{D}_0 := S - C_0$*), the conservative (resp. dissipative) part of* $\{\phi_t\}_{t \in \mathbb{Z}^d}$*.*

Suppose $r_t(s) = w_t^{(0)}(s)$, $t \in \mathbb{R}^d$, $s \in S$. Then from the proof of previous proposition we get the following corollary, which is the continuous-parameter analogue of Corollary 2.2.4.

Corollary 4.2.2. *For any* $h \in L^1(S,\mu)$*,* $h > 0$*, the conservative part of* $\{\phi_t\}_{t \in \mathbb{R}^d}$ *is given by*

$$C = \{s \in S : \int_{\mathbb{R}^d} h \circ \phi_t(s) r_t(s) \lambda(dt) = \infty\} \quad mod\, \mu.$$

As in the discrete case, the action $\{\phi_t\}$ is called conservative if $S = C$ and dissipative if $S = \mathcal{D}$.

Remark 4.2.3. We know that the Radon-Nikodym derivatives $\{\frac{d\mu \circ \phi_t}{d\mu}\}$ induced by the group action $\{\phi_t\}$ satisfy the equations

$$\frac{d\mu \circ \phi_{t_1+t_2}}{d\mu} = \frac{d\mu \circ \phi_{t_1}}{d\mu} \times \frac{d\mu \circ \phi_{t_2}}{d\mu} \circ \phi_{t_1} \quad \text{for all } t_1 \text{ and } t_2$$

except on a set of measure zero. However, this exceptional set depends on t_1 and t_2, which are both uncountably many. This may lead us to technical difficulties because we will use the above equation quite a few times in the next section. In order to resolve this problem, $\{r_t\}$ is constructed carefully here using the result of Varadarajan (1970). As we will see, $\{r_t\}$ will play a very significant role in what follows.

4.3 Structure of Stationary SαS Random Fields

Suppose $\mathbf{X} = \{X_t\}_{t\in\mathbb{R}^d}$ is a measurable $S\alpha S$ random field, $0 < \alpha < 2$. Theorem 13.2.1 in Samorodnitsky and Taqqu (1994) implies that $\mathbf{X}$ has an integral representation of the from

$$X_t \stackrel{d}{=} \int_S f_t(s)M(ds), \quad t \in \mathbb{R}^d\,, \tag{4.3.1}$$

where M is a $S\alpha S$ random measure on some standard Borel space $(S, \mathcal{S})$ with σ-finite control measure μ and $f_t \in L^\alpha(S, \mu)$ for all $t \in \mathbb{R}^d$. Since the random field is assumed to be measurable, by Theorem 11.1.1 in Samorodnitsky and Taqqu (1994)) we can always choose the kernel $\{f_t\}$ in such a way that $(t, s) \mapsto f_t(s)$ is jointly measurable on $\mathbb{R}^d \times S$. Such an integral representation is called a measurable representation of $\mathbf{X}$. As in the discrete-parameter case, f_t's are deterministic functions and hence all the randomness of $\mathbf{X}$ is hidden in the random measure M, and the inter-dependence of the X_t's is captured in $\{f_t\}$. As before, we can again assume, without loss of generality, that the family $\{f_t\}$ satisfies the full support assumption

$$\text{Support}\Big(f_t,\ t \in \mathbb{R}^d\Big) = S\ . \tag{4.3.2}$$

As in the discrete-parameter case the integral representation takes of a special form provided $\{X_t\}$ is stationary. See, once again, Rosiński (1995) for the $d = 1$ case and

Rosiński (2000) for a general d. Specifically, every measurable minimal representation (this can be defined in parallel to Definition 2.3.1 and such a representation exists by Theorem 2.2 in Rosiński (1995)) of $\mathbf{X}$ turns out to be of the form

$$f_t(s) \quad = \quad c_t(s)\left(\frac{d\mu \circ \phi_t}{d\mu}(s)\right)^{1/\alpha} f \circ \phi_t(s), \quad t \in \mathbb{R}^d, \tag{4.3.3}$$

where $f \in L^\alpha(S, \mu)$, $\{\phi_t\}_{t \in \mathbb{R}^d}$ is a nonsingular $\mathbb{R}^d$-action on (S, μ) and $\{c_t\}_{t \in \mathbb{R}^d}$ is a measurable cocycle for $\{\phi_t\}$ taking values in $\{-1, +1\}$ i.e., $(t, s) \mapsto c_t(s)$ is a jointly measurable map $\mathbb{R}^d \times S \to \{-1, +1\}$ such that for all $u, v \in \mathbb{R}^d$

$$c_{u+v}(s) = c_v(s) c_u(\phi_v(s)) \ \text{ for } \mu\text{-a.a. } s \in S. \tag{4.3.4}$$

Conversely, if $\{f_t\}$ is of the form (4.3.3) then $\{X_t\}$ defined by (4.3.1) is a stationary $S\alpha S$ random field.

Remark 4.3.1. We can always choose the cocycle $\{c_t\}$ in (4.3.3) in such way that (4.3.4) holds for all $(u, v, s) \in \mathbb{R}^d \times \mathbb{R}^d \times S$. This follows by Appendix B9 in Zimmer (1984).

In order to study the measurable stationary $S\alpha S$ random fields we first establish that any measurable stationary random field indexed by $\mathbb{R}^d$ is continuous in probability. In the one-dimensional case, the corresponding result for measurable processes with stationary increments was proved by Surgailis et al. (1998).

Proposition 4.3.2. *Suppose* $\mathbf{X} = \{X_t\}_{t \in \mathbb{R}^d}$ *be a measurable stationary random field. Then* $\mathbf{X}$ *is continuous in probability i.e., for every* $t_0 \in \mathbb{R}^d$, $X_t \xrightarrow{p} X_{t_0}$ *whenever* $t \to t_0$.

Proof. It is enough to show $\{X_t^M\}_{t \in \mathbb{R}^d}$ is continuous in probability for all $M > 0$, where

$$X_t^M = \begin{cases} X_t & \text{if } |X_t| \leq M, \\ 0 & \text{otherwise.} \end{cases}$$

Hence, without loss of generality, we can assume that $\|X_0\|_2 < \infty$ where $\|\cdot\|_2$ denotes the L^2-norm.

Let $(\Omega, \mathcal{A}, P)$ be the underlying probability space, and Σ be the path space

$$\Sigma := \{(X_t(\omega),\ t \in \mathbb{R}^d) : \omega \in \Omega\}\,.$$

Define $\{\phi_t\}_{t\in\mathbb{R}^d}$ on Σ as follows:

$$\phi_t(u)(s) = u(s+t) \quad \text{for all } u \in \Sigma\,.$$

By measurability of $\mathbf{X}$ it follows that $\{\phi_t\}$ is an $\mathbb{R}^d$-action. Define σ to be the induced probability measure on the path space Σ, namely

$$\sigma(A) = P(\{\omega : \ (X_t(\omega),\ t \in \mathbb{R}^d) \in A\}).$$

Stationarity of $\mathbf{X}$ implies that $\{\phi_t\}$ preserves σ. For all $t \in \mathbb{R}^d$, define a random variable Y_t on Σ as

$$Y_t(u) = u(t) \text{ for all } u \in \Sigma\,.$$

Note that

$$Y_t = \sqrt{\frac{d\sigma \circ \phi_t}{d\sigma}}\, Y_0 \circ \phi_t \text{ for all } t \in \mathbb{R}^d$$

and by our assumption $Y_0 \in L^2(\Sigma, \sigma)$. Hence using Banach's theorem for Polish groups (see Section 1.6 in Aaronson (1997)) it follows that $t \mapsto Y_t$ is L^2-continuous, which is same as saying $t \mapsto X_t$ is L^2-continuous, which implies the result. $\square$

As is the discrete-parameter case, we say that a measurable stationary $S\alpha S$ random field $\{X_t\}_{t\in\mathbb{R}^d}$ is generated by a nonsingular $\mathbb{R}^d$-action $\{\phi_t\}$ on (S,μ) if it has an integral representation of the form (4.3.3) satisfying (4.3.2). The following result, which is the continuous-parameter analogue of Proposition 2.3.3, yields that the classes of measurable stationary $S\alpha S$ random fields generated by conservative and dissipative actions are disjoint.

Let $T_0 \in \mathcal{B}$, $T_0 \subseteq F_0 = [0, \mathbf{1})$ be such that $\mu(F_0 - T_0) = 0$ and for all $t \in T_0,\ s \mapsto w_t(s)$ is a version of $\frac{d\mu\circ\phi_t}{d\mu}$ where $\{w_t(s)\}$ is as in Section 4.2. Let $T = \bigcup_{\gamma\in\Gamma_0}(\gamma + T_0)$. Then for all $t \in T$, $s \mapsto r_t(s)$ is a version of $\frac{d\mu\circ\phi_t}{d\mu}$. Here $\{r_t(s)\}$ is also as defined in Section 4.2. Define, for $t \in \mathbb{R}^d$,

$$\tilde{f}_t(s) = c_t(s) f \circ \phi_t(s)(r_t(s))^{1/\alpha}\,, \quad s \in S\,. \tag{4.3.5}$$

Proposition 4.3.3. *Suppose $\{X_t\}_{t\in\mathbb{R}^d}$ is a measurable stationary $S\alpha S$ random field generated by a nonsingular $\mathbb{R}^d$-action $\{\phi_t\}$ on (S,μ) and $\{f_t\}$ is given by* (4.3.3)*. Also let C and $\mathcal{D}$ be the conservative and dissipative parts of $\{\phi_t\}$. Then we have*

$$\begin{aligned} C &= \{s \in S : \int_{\mathbb{R}^d} |\tilde{f}_t(s)|^\alpha = \infty\} \ \ mod\, \mu, \text{ and} \\ \mathcal{D} &= \{s \in S : \int_{\mathbb{R}^d} |\tilde{f}_t(s)|^\alpha < \infty\} \ \ mod\, \mu\,, \end{aligned}$$

where $\{\tilde{f}_t\}$ is as in (4.3.5)*. In particular, if a stationary $S\alpha S$ random field $\{X_t\}_{t\in\mathbb{R}^d}$ is generated by a conservative (dissipative, resp.) $\mathbb{R}^d$-action, then in any other integral representation of $\{X_t\}$ of the form* (4.3.3) *satisfying* (4.3.2)*, the $\mathbb{R}^d$-action must be conservative (dissipative, resp.).*

Proof. Since $\{X_t\}$ is stationary and measurable, we have $\{X_t\}$ is continuous in probability which implies $\{f_t\}$ is L^α-continuous. Hence using (4.3.2) and the fact that T is dense in $\mathbb{R}^d$ it follows that

$$Support\{\tilde{f}_t : t \in T\} = S \tag{4.3.6}$$

from which we get by Fubini's Theorem that

$$\int_T |\tilde{f}_t(s)|^\alpha \lambda(dt) > 0 \quad \text{for } \mu\text{-a.a. } s \in S\,. \tag{4.3.7}$$

Let

$$h(s) = \sum_{\gamma\in\mathbb{Z}^d} a_\gamma \int_{\gamma+T_0} |\tilde{f}_t(s)|^\alpha d\lambda(t)\,, \quad s \in S\,,$$

where $a_\gamma > 0$ for all $\gamma \in \mathbb{Z}^d$ and $\sum_{\gamma\in\mathbb{Z}^d} a_\gamma = 1$. Clearly, $h \in L^1(S,\mu)$ and by (4.3.7), $h > 0$. Let

$$S' := \{s \in S : w_{\gamma_1+\gamma_2}(s) = w_{\gamma_1} \circ \phi_{\gamma_2}(s)\, w_{\gamma_2}(s) \text{ for all } \gamma_1, \gamma_2 \in \mathbb{Z}^d\}\,.$$

Then $\mu(S - S') = 0$ and for all $s \in S'$ we have

$$\begin{aligned}
&\sum_{\beta\in\mathbb{Z}^d} h \circ \phi_\beta(s) r_\beta(s) \\
&= \sum_{\beta\in\mathbb{Z}^d} \sum_{\gamma\in\mathbb{Z}^d} a_\gamma \int_{\gamma+T_0} \left|\tilde{f}_t(\phi_\beta(s))\right|^\alpha r_\beta(s) \lambda(dt) \\
&= \sum_{\beta\in\mathbb{Z}^d} \sum_{\gamma\in\mathbb{Z}^d} a_\gamma \int_{\gamma+T_0} \left|f(\phi_{t+\beta}(s))\right|^\alpha r_t(\phi_\beta(s)) r_\beta(s) \lambda(dt) \\
&= \sum_{\beta\in\mathbb{Z}^d} \sum_{\gamma\in\mathbb{Z}^d} a_\gamma \int_{T_0} \left|f(\phi_{t+\beta+\gamma}(s))\right|^\alpha r_{t+\gamma}(\phi_\beta(s)) r_\beta(s) \lambda(dt)\,,
\end{aligned}$$

the last step following by translation invariance of λ. Using the definition of S' and (4.2.3) for $m = 0$ this equals

$$\begin{aligned}
&= \sum_{\beta\in\mathbb{Z}^d} \sum_{\gamma\in\mathbb{Z}^d} a_\gamma \int_{T_0} \left|f(\phi_{t+\beta+\gamma}(s))\right|^\alpha r_t(\phi_{\beta+\gamma}(s)) r_\gamma(\phi_\beta(s)) r_\beta(s) \lambda(dt) \\
&= \sum_{\gamma\in\mathbb{Z}^d} a_\gamma \sum_{\beta\in\mathbb{Z}^d} \int_{T_0} \left|f(\phi_{t+\beta+\gamma}(s))\right|^\alpha r_t(\phi_{\beta+\gamma}(s)) r_{\beta+\gamma}(s) \lambda(dt) \\
&= \sum_{\beta\in\mathbb{Z}^d} \int_{T_0} \left|f(\phi_{t+\beta}(s))\right|^\alpha r_t(\phi_\beta(s)) r_\beta(s) \lambda(dt) \\
&= \sum_{\beta\in\mathbb{Z}^d} \int_{T_0} \left|f(\phi_{t+\beta}(s))\right|^\alpha r_{t+\beta}(s) \lambda(dt)\,,
\end{aligned}$$

which, by another use of translation invariance of λ, becomes

$$\begin{aligned}
&= \sum_{\beta\in\mathbb{Z}^d} \int_{\beta+T_0} \left|f(\phi_t(s))\right|^\alpha r_t(s) \lambda(dt) \\
&= \int_T |\tilde{f}_t(s)|^\alpha \lambda(dt) = \int_{\mathbb{R}^d} |\tilde{f}_t(s)|^\alpha \lambda(dt)\,.
\end{aligned}$$

Hence, by Corollary 2.2.4, we have

$$\begin{aligned}
C = C_0 &= \{s \in S : \sum_{\beta\in\mathbb{Z}^d} h \circ \phi_\beta(s) r_\beta(s) = \infty\} \\
&= \{s \in S : \int_{\mathbb{R}^d} |\tilde{f}_t(s)|^\alpha \lambda(dt) = \infty\} \mod \mu\,.
\end{aligned}$$

This completes the proof of the first part.

To prove the second part, let $\{\psi_t\}$ be a $\mathbb{R}^d$-action on (Y, ν) which also generates $\{X_t\}_{t\in\mathbb{R}^d}$. This means

$$g_t = u_t \left(\frac{d\nu \circ \psi_t}{d\nu}\right)^{1/\alpha} g \circ \psi_t\,, \quad t \in \mathbb{R}^d$$

is another representation of $\{X_t\}$ satisfying the full support condition (4.3.2) where $g \in L^\alpha(Y, \nu)$, and $\{u_t\}_{t\in\mathbb{R}^d}$ is a measurable cocycle for $\{\psi_t\}$. We have to show $\{\psi_t\}$ is conservative as well. By the first part, it is enough to show that

$$\int_{\mathbb{R}^d} |\tilde{g}_t(s)|^\alpha \lambda(dt) = \infty \quad \text{for } \mu\text{-a.a. } s \in S, \tag{4.3.8}$$

where for all $t \in \mathbb{R}^d$,

$$\tilde{g}_t(s) = u_t(s) g \circ \psi_t(s) (q_t(s))^{1/\alpha}\,, \quad s \in S\ .$$

and $\{q_t\}$ is constructed for $\{\psi_t\}$ following verbatim the construction of $\{r_t\}$ in Section 4.2.

Once again using L^α-continuity of $\{g_t\}$ we can establish

$$Support\{\tilde{g}_t : t \in T\} = S\ . \tag{4.3.9}$$

Notice that both $\{\tilde{f}_t\}_{t\in T}$ and $\{\tilde{g}_t\}_{t\in T}$ are integral representations of the subfield $\{X_t\}_{t\in T}$ and $\{\tilde{g}_t\}$ satisfies the full support condition (4.3.9). Hence (4.3.8) follows by Theorem 1.1 in Rosiński (1995) and an argument parallel to the discrete parameter case. □

The following continuous-parameter analogue of Corollary 2.3.4 states that the test described in the previous proposition can be applied to any integral representation of the random field as long as it has full support.

Corollary 4.3.4. *The measurable stationary $S\alpha S$ random field $\{X_t\}_{t\in\mathbb{R}^d}$ is generated by a conservative (dissipative, resp.) $\mathbb{R}^d$-action if and only if for any (equivalently, some) measurable representation* (4.3.1) *of $\{X_t\}$ satisfying* (4.3.2)*, the integral*

$$\int_{\mathbb{R}^d} |f_t(s)|^\alpha d\lambda(t) \tag{4.3.10}$$

is infinite (finite, resp) μ-a.e. .

Proof. Fix a measurable minimal representation $\{f_t^{(1)}\}_{t\in\mathbb{R}^d}$ of $\{X_t\}_{t\in\mathbb{R}^d}$. Define $\{\tilde{f}_t^{(1)}\}_{t\in\mathbb{R}^d}$ in parallel to (4.3.5). Taking a minimal representation $\{g_t\}_{t\in T}$ of the subfield $\{X_t\}_{t\in T}$ we observe once again that both the integral representations $\{\tilde{f}_t^{(1)}\}_{t\in T}$ and $\{g_t\}_{t\in T}$ of the subfield $\{X_t\}_{t\in T}$ satisfy the full support condition. Hence we can use Remark 2.5 in Rosiński (1995) together with the arguments given in the discrete-parameter case twice to conclude

$$\begin{aligned}
\int_{\mathbb{R}^d} |f_t|^\alpha \lambda(dt) < \infty \quad \mu\text{-a.e.} &\iff \int_T |f_t|^\alpha \lambda(dt) < \infty \quad \mu\text{-a.e.} \\
&\iff \int_T |g_t|^\alpha \lambda(dt) < \infty \quad \mu\text{-a.e.} \\
&\iff \int_T \left|\tilde{f}_t^{(1)}\right|^\alpha \lambda(dt) < \infty \quad \mu\text{-a.e.} \\
&\iff \int_{\mathbb{R}^d} \left|\tilde{f}_t^{(1)}\right|^\alpha \lambda(dt) < \infty \quad \mu\text{-a.e.}.
\end{aligned}$$

and since $\{f_t^{(1)}\}_{t\in\mathbb{R}^d}$ is of the form (4.3.3) and satisfies (4.3.2), this corollary follows from Proposition 4.3.3. □

As in the discrete-parameter case, Proposition 4.3.3 enables us to connect the decomposition of a stable random field into three independent parts available in Rosiński (2000) to the conservative-dissipative decomposition of the underlying action. For the continuous-parameter case, mixed moving average can be defined in parallel to (2.3.7) as follows:

$$\mathbf{X} \stackrel{d}{=} \left\{ \int_{W\times\mathbb{R}^d} f(v, t+s)\, M(dv, ds) \right\}_{t\in\mathbb{R}^d}, \tag{4.3.11}$$

where $f \in L^\alpha(W\times\mathbb{R}^d, \nu\otimes\lambda)$, λ is the Lebesgue measure on $\mathbb{R}^d$, ν is a σ-finite measure on a standard Borel space $(W, \mathcal{W})$, and the control measure μ of M equals $\nu\otimes\lambda$ (see, once again, Surgailis et al. (1993) and Rosiński (2000)). The following result gives three equivalent characterizations of stationary $S\alpha S$ random fields generated by dissipative actions.

Theorem 4.3.5. *Suppose $\{X_t\}_{t \in \mathbb{R}^d}$ is a stationary $S\alpha S$ random field. Then, the following are equivalent:*

1. *$\{X_t\}$ is generated by a dissipative $\mathbb{R}^d$-action.*
2. *For any measurable representation $\{f_t\}$ of $\{X_t\}$ we have*
$$\int_{\mathbb{R}^d} |f_t(s)|^\alpha < \infty \text{ for } \mu\text{-a.a. } s.$$
3. *$\{X_t\}$ is a mixed moving average.*
4. *$\{X_t\}_{t \in \Gamma_n}$ is a mixed moving average for some (all) $n \geq 1$.*

Proof. 1 and 2 are equivalent by Corollary 4.3.4, and 2 and 3 are equivalent by Theorem 2.1 of Rosiński (2000). 1 and 4 are equivalent by Theorem 2.3.5. □

Therefore, in order to establish that $\mathbf{X}$ is a mixed moving average, it is enough to look at one of its discrete skeleton. Theorem 4.3.5 also allows us to describe the decomposition of a stationary $S\alpha S$ random field given in Theorem 3.7 of Rosiński (2000) in terms of the ergodic theoretical properties of nonsingular $\mathbb{R}^d$-actions generating the field. The statement of the following corollary is an extension of the one-dimensional decomposition in Theorem 4.3 in Rosiński (1995) to random fields.

Corollary 4.3.6. *A stationary $S\alpha S$ random field $\mathbf{X}$ has a unique in law decomposition*

$$X_t \stackrel{d}{=} X_t^{\mathcal{C}} + X_t^{\mathcal{D}}, \tag{4.3.12}$$

where $\mathbf{X}^{\mathcal{C}}$ and $\mathbf{X}^{\mathcal{D}}$ are two independent stationary $S\alpha S$ random fields such that $\mathbf{X}^{\mathcal{D}}$ is a mixed moving average, and $\mathbf{X}^{\mathcal{C}}$ is generated by a conservative action.

As in the discrete-parameter case, one can think of stable random fields generated by conservative actions as having longer memory than those generated by dissipative actions for the exact same heuristic reason.

4.4 Rate of Growth of the Maxima

As in the discrete-parameter case, the extreme values of $\{X_t\}$ tend to grow at a slower rate if $\{X_t\}$ is generated by a conservative action. For $d = 1$, this has been formalized in Samorodnitsky (2004b), and in this section we will see that it turns out to be the case for stable random fields as well. In order to extend the one-dimensional result on extremes to higher dimensions we need to assume that $\mathbf{X} = \{X_t\}_{t\in\mathbb{R}^d}$ is locally bounded apart from being stationary and measurable. If further $\mathbf{X}$ is separable then

$$M_\tau = \sup_{0\le s\le \tau\mathbf{1}} |X_s|, \quad \tau \ge 0, \tag{4.4.1}$$

is a well-defined finite-valued stochastic process. Here,

$$u = (u^{(1)}, u^{(2)}, \ldots, u^{(d)}) \le v = (v^{(1)}, v^{(2)}, \ldots, v^{(d)})$$

means $u^{(i)} \le v^{(i)}$ for all $i = 1, 2, \ldots, d$ and $\mathbf{1} := (1, 1, \ldots, 1) \in \mathbb{R}^d$. Since $\mathbf{X}$ is stationary and measurable, it is continuous in probability by Proposition 4.3.2. Therefore, taking its separable version the above maxima process can be defined by

$$M_\tau = \sup_{s\in[0,\,\tau\mathbf{1}]\cap\Gamma} |X_s|, \quad \tau \ge 0, \tag{4.4.2}$$

where $\Gamma := \bigcup_{n=1}^\infty \Gamma_n = \bigcup_{n=1}^\infty \frac{1}{2^n}\mathbb{Z}^d$ and $[u, v] := \{s \in \mathbb{R}^d : \ u \le s \le v\}$. This will avoid the usual measurability problems of the uncountable maximum (4.4.1). Another advantage of this method is that a lot of uncountable suprema of functions can be treated in the same way without bothering about the measurability issues. For instance, we write $\sup_{t\in A} |f_t(x)|$ for some set $A \subseteq \mathbb{R}^d$, to mean the measurable function $\sup_{t\in A\cap\Gamma} |f_t(x)|$.

Keeping the above discussion in mind we define, for $\tau \ge 0$,

$$b_\tau := \left(\int_S \sup_{0\le t\le \tau\mathbf{1}} |f_t(s)|^\alpha \mu(ds) \right)^{1/\alpha}. \tag{4.4.3}$$

in parallel to (2.5.2). By local boundedness of $\mathbf{X}$ and Theorem 10.2.3 of Samorodnitsky and Taqqu (1994) it follows that $b_\tau < \infty$ for all $\tau \ge 0$. Also, Corollary 4.4.6 in

Samorodnitsky and Taqqu (1994) implies that b_τ does not depend on the choice of integral representation of $\mathbf{X}$. As in the discrete-parameter case the rate of growth of the maxima process depends heavily on the rate of growth of b_τ.

In order to estimate the rate of growth of the maxima process (4.4.2) we first do the same for the the deterministic function (4.4.3). As in the discrete-parameter case, the asymptotic behavior of b_τ depends heavily on the ergodic theoretic properties of the underlying nonsingular $\mathbb{R}^d$-action.

Proposition 4.4.1. *(i) If the action $\{\phi_t\}$ is conservative, then*

$$\tau^{-d/\alpha} b_\tau \to 0 \;\; \textit{as } \tau \to \infty. \tag{4.4.4}$$

(ii) If the action $\{\phi_t\}$ is dissipative, then

$$\lim_{\tau\to\infty} \tau^{-d/\alpha} b_\tau = \left(\int_W g(v)^\alpha \nu(dv) \right)^{1/\alpha} \in (0, \infty), \tag{4.4.5}$$

where we can use any mixed moving average representation (4.3.11) *of the random field and*

$$g(v) = \sup_{s\in\mathbb{R}^d} |f(v,s)| \;\; \textit{for } v \in W.$$

Proof. (i) Define, for $n = 1, 2, 3, \ldots$, and for $k \in \{2^j : j = 0, 1, 2, \ldots\}$,

$$U_{k,n} := \left\{ u \in \frac{1}{k}\mathbb{Z}^d : u \geq 0, \; \|u\|_\infty \leq n \right\} .$$

Let $\epsilon > 0$. Since $b_1 < \infty$, by monotone convergence theorem we can find a $k \in \{2^j : j = 0, 1, 2, \ldots\}$ such that

$$\int_S \max_{u\in U_{k,1}} |f_u(s)|^\alpha \mu(ds) \geq b_1 - \epsilon \,. \tag{4.4.6}$$

Since $\{\phi_t\}_{t\in\mathbb{R}^d}$ is conservative, so is the $\mathbb{Z}^d$-action

$$\psi_u(s) := \phi_{u/k}(s), \quad u \in \mathbb{Z}^d \,,$$

which generates the discrete-parameter stationary $S\alpha S$ random field

$$Y_u = X(u/k) = \int_S f_{u/k}(s)M(ds), \quad u \in \mathbb{Z}^d .$$

Hence, by Proposition 2.5.1, we obtain that

$$\lim_{n\to\infty} n^{-d/\alpha}\tilde{b}_n = 0\,, \tag{4.4.7}$$

where

$$\tilde{b}_n := \Big(\int_S \max_{u\in U_{k,n}} |f_u(s)|^\alpha \mu(ds)\Big)^{1/\alpha}, \quad n = 0, 1, 2, \ldots .$$

Following verbatim the argument presented in the one-dimensional case Samorodnitsky (2004b) we have for $N = 1, 2, 3, \ldots$,

$$\begin{aligned}\int_S \sup_{0\le t\le N\mathbf{1}} |f_t(s)|^\alpha \mu(ds) &\le \tilde{b}_{Nk}^\alpha + N^d\Big(b_1 - \int_S \max_{u\in U_{k,1}} |f_u(s)|^\alpha \mu(ds)\Big) \\ &\le \tilde{b}_{Nk}^\alpha + N^d\epsilon\,.\end{aligned}$$

This yields

$$b_\tau^\alpha \le b_{\lceil\tau\rceil}^\alpha \le \tilde{b}_{k\lceil\tau\rceil}^\alpha + (\tau+1)^d\epsilon,$$

from which (4.4.4) follows by (4.4.7).

(ii) For any mixed moving average representation (4.3.11) we have

$$\begin{aligned}\int_W g(v)^\alpha \nu(dv) &\le \sum_{u\in\mathbb{Z}^d}\int_W \sup_{u\le s\le u+\mathbf{1}} |f(v,s)|^\alpha \nu(dv) \\ &\le \sum_{u\in\mathbb{Z}^d}\int_W \Big(\int_{[u-\mathbf{1},u]} \sup_{0\le t\le \mathbf{2}} |f(v,s+t)|^\alpha ds\Big)\nu(dv) \\ &= \int_W\int_{\mathbb{R}^d} \sup_{0\le t\le \mathbf{2}} |f(v,s+t)|^\alpha ds\,\nu(dv) = b_2^\alpha < \infty\end{aligned}$$

where $\mathbf{2} := (2, 2, \ldots, 2)$. The proof of (4.4.5) is exactly same as the corresponding statement in the discrete-parameter case. One uses a direct computation to check the claim in the case where f has compact support, that is

$$f(v,s) = 0 \quad \text{for all } (v,s) \text{ with } \|s\|_\infty > A \ \text{ for some } A > 0.$$

The proof in the general case follows then by approximating a general kernel f by a kernel with a compact support. $\square$

At this point let us recall a series representation of the subfield $\{X_t\}_{0 \le t \le \tau\mathbf{1}}$ given by

$$X_t \stackrel{d}{=} b_\tau C_\alpha^{1/\alpha} \sum_{j=1}^{\infty} \varepsilon_j T_j^{-1/\alpha} \frac{|f_t(U_j^{(\tau)})|}{\sup_{0 \le u \le \tau\mathbf{1}} |f_u(U_j^{(\tau)})|}, \qquad t \in [0, \tau\mathbf{1}], \tag{4.4.8}$$

where $\varepsilon_1, \varepsilon_2, \ldots$ are i.i.d. Rademacher random variables (i.e., $P(\varepsilon_1 = 1) = P(\varepsilon_1 = -1) = 1/2$), $T_1, T_2, \ldots$ is a sequence of the arrival times of a unit rate Poisson process on $(0, \infty)$, $\{U_j^{(\tau)}\}$ are i.i.d. S-valued random variables with common law η_τ given by

$$\frac{d\eta_\tau}{d\mu}(s) = b_\tau^{-\alpha} \sup_{0 \le t \le \tau\mathbf{1}} |f_t(s)|^\alpha, \qquad s \in S, \tag{4.4.9}$$

and all three sequences above are independent; see, for example, Samorodnitsky and Taqqu (1994) for details.

The next result is the continuous-parameter analogue of Theorem 2.6.1. It's proof is based two main ingredients, namely, Proposition 4.4.1, and the series representation (4.4.8), and the argument is exactly same as the one-dimensional case (Theorem 2.2 in Samorodnitsky (2004b)).

Theorem 4.4.2. *Let* $\mathbf{X} = \{X_t\}_{t \in \mathbb{R}^d}$ *be a stationary, locally bounded* $S\alpha S$ *random field, where* $0 < \alpha < 2$.

(i) Suppose that $\mathbf{X}$ *is not generated by a conservative action (i.e., the component* $X^{\mathcal{D}}$ *in* (4.3.12) *generated by the dissipative part is nonzero). Then*

$$\frac{1}{\tau^{d/\alpha}} M_\tau \Rightarrow C_\alpha^{1/\alpha} K_X Z_\alpha \tag{4.4.10}$$

as $\tau \to \infty$, *where*

$$K_X = \left(\int_W (g(v))^\alpha \nu(dv) \right)^{1/\alpha}$$

and g is given by (4.4.1) *for any representation of $X^{\mathcal{D}}$ in the mixed moving average form* (4.3.11), *C_α is the stable tail constant (see (1.2.9) in Samorodnitsky and Taqqu (1994)) and Z_α is the standard Frechet-type extreme value random variable with the distribution*

$$P(Z_\alpha \le z) = e^{-z^{-\alpha}}, \quad z > 0.$$

(ii) Suppose that $\mathbf{X}$ is generated by a conservative $\mathbb{R}^d$-action. Then

$$\frac{1}{\tau^{d/\alpha}} M_\tau \xrightarrow{p} 0 \tag{4.4.11}$$

as $\tau \to \infty$. Furthermore, with b_τ given by (4.4.3),

$$\left\{ \frac{1}{c_\tau} M_\tau \right\} \text{ is not tight for any positive function} c_\tau = o(b_\tau), \tag{4.4.12}$$

while if, for some $\theta > 0$ and $c > 0$,

$$b_\tau \ge c\tau^\theta \quad \text{for all } \tau \text{ large enough} \tag{4.4.13}$$

then

$$\left\{ \frac{1}{b_\tau} M_\tau \right\} \text{ is tight.} \tag{4.4.14}$$

Finally, for $\tau > 0$, let $\{U_j^{(\tau)}\}_{j \ge 1}$ be as in (4.4.8)*. Suppose that* (4.4.13) *holds and for any $\epsilon > 0$,*

$$P\Bigg(\text{for some } t \in [0, \tau\mathbf{1}], \\ \frac{|f_t(U_j^{(\tau)})|}{\sup_{0 \le u \le \tau\mathbf{1}} |f_u(U_j^{(\tau)})|} > \epsilon, \ j = 1, 2\Bigg) \to 0 \tag{4.4.15}$$

as $\tau \to \infty$. Then

$$\frac{1}{b_\tau} M_\tau \Rightarrow C_\alpha^{1/\alpha} Z_\alpha \tag{4.4.16}$$

as $\tau \to \infty$. A sufficient condition for (4.4.15) *is*

$$\lim_{\tau \to \infty} \frac{b_\tau}{\tau^{d/2\alpha}} = \infty. \tag{4.4.17}$$

Remark 4.4.3. Unlike the discrete-parameter case, we cannot give a better estimate of the rate of growth of the maxima when the underlying action is conservative. In general, this rate depends on the action as well as on the kernel f.

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